Height representation of XOR-Ising loops
via bipartite dimers

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Abstract

The XOR-Ising model on a graph consists of random spin configurations on vertices of the graph obtained by taking the product at each vertex of the spins of two independent Ising models. In this paper, we explicitly relate loop configurations of the XOR-Ising model and those of a dimer model living on a decorated, bipartite version of the Ising graph. This result is proved for graphs embedded in compact surfaces of genus g. Using this fact, we then prove that XOR-Ising loops have the same law as level lines of the height function of this bipartite dimer model. At criticality, the height function is known to converge weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field [dT07b]. As a consequence, results of this paper shed a light on the occurrence of the Gaussian free field in the XOR-Ising model. In particular, they provide a step forward in the solution of Wilson's conjecture [Wil11], stating that the scaling limit of XOR-Ising loops are level lines of the Gaussian free field.

1   Introduction

The *double Ising model* consists of two Ising models, living on the same graph. It is related [KW71, Wu71, Fan72, Weg72] to other models of statistical mechanics, as the 8-vertex model [Sut70, FW70] and the Ashkin-Teller model [AT43]. In general, the two models maybe interacting. However, in this paper, we consider the case of two non-interacting Ising models, defined on the dual $G^* = (V^*, E^*)$ of a graph $G = (V, E)$, having the same coupling constants $(J_{e^*})_{e^* \in E^*}$, where the graph G is embedded either in a compact, orientable surface Σ of genus $g \geq 0$, or in the plane.

We are interested in the polarization of the model [KB79], also referred to as the XOR-*Ising model* [Wil11] by Wilson. It is defined as follows: given a pair of spin configurations $(\sigma, \sigma') \in \{-1, 1\}^{V^*} \times \{-1, 1\}^{V^*}$, the XOR-*spin configuration* belongs to $\{-1, 1\}^{V^*}$ and is obtained by taking, at every vertex, the product of the spins. The interface between ± 1 spin configurations of the XOR-configuration is a loop configuration of the graph G. Using

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extensive simulations, Wilson [Wil11] finds that, when G is a specific simply connected domain of the plane, and when both Ising models are critical, XOR loop configurations are contour lines of the Gaussian free field, with heights of the contours spaced $\sqrt{2}$ times as far apart as they should be for the double dimer model on the square lattice. Similar conjectures, involving SLE rather than the Gaussian free field, are obtained through conformal field theory [IR11, PS11]. Results of this paper give a partial proof of the occurence of the Gaussian free field.

The first part of the paper concentrates on finite graphs embedded in surfaces. We explicitly relate XOR loop configurations to loop configurations in a bipartite dimer model, implying in particular that both loop configurations have the same probability distribution. In the second part, we prove that this correspondence still holds for a large class of infinite planar graphs, the so-called *isoradial graphs* [Ken02, KS05], at criticality, and make the connection with Wilson's conjecture. Here is an outline.

Outline

Section 2. One of the tools required is a version of Kramers and Wannier's low/high temperature duality [KW41a, KW41b] in the case of graphs embedded in surfaces of genus g, *with boundary*. In the literature, we did find versions of this duality for graphs embedded in surfaces of genus g [LG94], but we could not find versions taking into account boundaries. This is the subject of Propositions 6 and 7, it involves relative homology theory and the Poincaré-Lefschetz duality.

Sections 3 and 4 consist in the extension to general graph embedded in a surface of genus g of an expansion due to Nienhuis [Nie84], which can be summarized as follows. Consider the low temperature expansion of the double Ising model, *i.e.* consider pairs of polygon configurations separating clusters of ± 1 spins of each spin configuration. Drawing both polygon configurations on G yields an edge configuration consisting of *monochromatic edges*, that is edges covered by exactly one of the two polygon configurations, and *bichromatic edges*, that is edges covered by both polygon configurations. Monochromatic edge configurations exactly correspond to XOR loop configurations, and separate the surface Σ into connected components $\Sigma_1, \cdots, \Sigma_N$. Inside each connected component, the law of bichromatic edge configurations is that of the low temperature expansion of an Ising model with coupling constants that are doubled. As a consequence, the partition function of the double Ising model can be rewritten using XOR loop configurations and bichromatic edge configurations, see Proposition 10.

Fixing a monochromatic edge configuration, and applying low/high temperature duality to the single Ising model corresponding to bichromatic edges, yields a rewriting of the double Ising partition function, as a sum over pairs of non intersecting polygon configurations of the primal and dual graph, where primal polygon configurations exactly correspond to XOR loop configurations, see Proposition 12 and Corollary 13. Note that there are quite a few difficulties in the proofs, due to the fact that we work on a surface of genus g.

Proposition 1.
The double Ising partition function for a graph embedded on a surface of genus g can be rewritten as:

$$Z_{\text{d-Ising}}(G^*, J) = \mathcal{C}_{\text{I}} \sum_{\substack{\{(P,P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*): \\ P \cap P^* = \emptyset\}}} \left(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}} \right) \left(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}} \right),$$

where primal polygon configurations of $\mathcal{P}^0(G)$ are the XOR loop configurations, and $\mathcal{C}_{\text{I}} = 2^{|V^*|+2g+1} \left(\prod_{e \in E} \cosh(2J_{e^*}) \right)$.

Section 5. In Section 5.1, we define the 6-vertex model on the medial graph G^{M} constructed from G. Reformulating an argument of Nienhuis [Nie84], we prove that the 6-vertex partition function can be written as a sum over non-intersecting pairs of polygon configurations of the primal and dual graph.

In Section 5.2, we define the dimer model on the decorated, bipartite graph G^{Q} constructed from G. Then, we present the mapping between dimer configurations of G^{Q} and *free fermionic* 6-vertex configurations of G^{M} [WL75, Dub11b]. Using both mappings, one assigns to every dimer configuration M a pair $\text{Poly}(M) = (\text{Poly}_1(M), \text{Poly}_2(M))$ of non intersecting primal and dual polygon configurations. The weights of the 6-vertex model chosen to match those of edges in the mixed contour expansion of the double Ising model satisfy the *free fermionic* condition. As a consequence, we then obtain, see also Proposition 17:

Proposition 2. *The dimer model partition function $Z_{\text{dimer}}^0(G^{\text{Q}}, J)$ can be rewritten as:*

$$Z_{\text{dimer}}^0(G^{\text{Q}}, J) = 2 \sum_{\substack{\{(P,P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*): \\ P \cap P^* = \emptyset\}}} \left(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}} \right) \left(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}} \right),$$

where primal and dual polygon configurations of $\mathcal{P}^0(G) \times \mathcal{P}^0(G^*)$ are the Poly configurations.

Combining Proposition 1 and Proposition 2 yields the following, see also Theorem 18:

Theorem 3. *XOR loop configurations of the double Ising model on G^* have the same law as Poly_1 configurations of the corresponding dimer model on the bipartite graph G^{Q}:*

$$\forall P \in \mathcal{P}^0(G), \quad \mathbb{P}_{\text{d-Ising}}[\text{XOR} = P] = \mathbb{P}_{\text{Q}}^0[\text{Poly}_1 = P].$$

Remark 4. In the paper [Dub11b], Dubédat relates a version of the double Ising model and the same bipartite dimer model in two ways. The first approach uses explicit mappings, most of which are present in the physics literature, and goes as follows. Consider a slightly different version of the double Ising model, with one model living on the primal graph G and the second on the dual graph G^*. This double Ising model can be mapped to an 8-vertex model [KW71, Wu71] on the medial graph. Using Fan and Wu's abelian duality,

this 8-vertex model [FW70] can be mapped to a second 8-vertex on the same graph. When coupling constants of the two Ising models satisfy Kramers and Wannier's duality, the second 8-vertex model is in fact a free-fermionic 6-vertex model. The free-fermionic 6-vertex model can in turn be mapped to a bipartite dimer model, a result due to [WL75] in the case of the square lattice, and extended by [Dub11b] in the general lattice case. It can also be seen as a specific case of the mapping of the free-fermionic 8-vertex model to a non-bipartite dimer model of [FW70]. Note that this bipartite dimer model is the model of *quadri-tilings* studied by the second author in [dT07a] and [dT07b].

When performing the different steps of the mapping, Dubédat keeps track of order/disorder variables, in the vein of [KC71]. Using results of a previous paper of his [Dub11a], this allows him to compute critical correlators in the plane. For simply connected regions, this result has independently been obtained by Chelkak, Hongler and Izyurov [CHI12].

Our goal here is different, since we aim at keeping track of XOR-configurations. This information is lost in the above approach. Indeed Fan and Wu's abelian duality for the 8-vertex model can be compared to a high temperature expansion, where configurations cannot be interpreted using the initial model.

The second approach uses transformation on matrices. The partition function of the double Ising model can be expressed using the determinant of the Kasteleyn matrix of the Fisher graph [Fis66]; whereas the partition function of the bipartite dimer model can be expressed using the Kasteleyn matrix of the graph G^Q. Dubédat shows that the two matrices are related through transformations not affecting the determinant. Using the fact that the partition function of the double Ising model is also related to the determinant of the Kac-Ward matrix [KW52], Cimasoni and Duminil-Copin use the same approach to relate the Kac-Ward matrix to the matrix of the same bipartite dimer model [CD12]. Their purpose is to identify the critical point of general bi-periodic Ising models, see also Li [Li10, Li12] for the case of the square lattice with arbitrary fundamental domain.

When performing the above transformation on matrices, one again looses track of double Ising configurations, and in particular of XOR-configurations.

Using Nienhuis' mapping [Nie84], the main contribution of this paper is to provide a coupling between the double Ising model and the bipartite dimer model, which *keeps track* of XOR loop configurations, and is valid for graphs embedded in surfaces of genus g.

Section 6. Suppose now that the two Ising models are critical and defined on the dual of an infinite isoradial graph G, see Section 6.1 for definitions. Then, the dimer model on the corresponding graph G^Q is also critical in the dimer sense. Using the locality property of both probability measures on Ising [BDT11], and dimer configurations [dT07b] on isoradial graphs at criticality, we prove that the equality in law stated in Theorem 3 still holds in this infinite context. See Theorem 20.

Section 7. The graph G^Q being bipartite, using a height function denoted h, dimer configurations can naturally be interpreted as discrete random interfaces. Our second theorem, see also Theorem 23, proves the following

Theorem 5. *XOR loop configurations of the double Ising model defined on G^* have the same law as level lines of the restriction of the height function h to vertices of the dual graph G^*.*

Theorem 5 can be seen as a proof of Wilson's conjecture mentioned above (see 7.2 for a precise statement) in the discrete setting, since it is known that the height function h, seen as a random distribution, converges in law in the scaling limit to $\frac{1}{\sqrt{\pi}}$ times the Gaussian free field in the plane [dT07b]. In particular, we explain in this section the special value of the spacing. It would yield a complete proof of the conjecture if we could overcome the same technical obstacles for the proof of the convergence of double dimer loops to CLE_4.

Acknowledgments: We would like to thank warmly Thierry Lévy for very helpful discussions on relative homology.

2 Ising model on graphs embedded in surfaces with boundary

In the whole of this section, we let G be a graph embedded in a compact orientable surface of genus g ($g \geq 0$), and G^* be its dual graph. The embedding of G^* is chosen such that dual vertices are in the interior of the corresponding faces.

Fix some positive integer p, and for every $i \in \{0, \cdots, p-1\}$, let B_i be a union of closed faces of G homeomorphic to a disc, such that $\bigcap_{i=0}^{p-1} B_i = \emptyset$. Denote by Σ the surface of genus g from which the union of the B_i's is removed. Then Σ is a compact orientable surface of genus g, with boundary $\partial\Sigma = \partial B_0 \cup \cdots \partial B_{p-1}$.

Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be the subgraph of G defined as follows: V_Σ consists of vertices of $V \cap \Sigma$; and E_Σ consists of edges of E joining vertices of V_Σ, from which edges on the boundary $\partial\Sigma$ are removed. Let $G_\Sigma^* = (V_\Sigma^*, E_\Sigma^*)$ be the subgraph of G^* whose vertices are vertices of $V^* \cap \Sigma$, and whose edges are edges of G^* joining vertices of V_Σ^*, see Figure 1 for an example. Note that the graph G_Σ^* contains all edges dual to edges of G_Σ, *i.e.* there is a bijection between primal edges of G_Σ and dual edges of G_Σ^*.

Fix a collection of positive constants $(J_{e^*})_{e^* \in E^*}$ attached to edges of G^*, referred to as *coupling constants*. The *Ising model on G_Σ^* with coupling constants* (J_{e^*}) is defined as follows. A *spin configuration* σ of G_Σ^* is a function of the vertices of V_Σ^* with values in $\{-1, +1\}$. The probability of occurrence of a spin configuration σ is given by the *Ising Boltzmann measure*, denoted $\mathbb{P}_{\mathrm{Ising}}$, and defined by:

$$\forall \sigma \in \{-1, 1\}^{V_\Sigma^*}, \quad \mathbb{P}_{\mathrm{Ising}}(\sigma) = \frac{1}{Z_{\mathrm{Ising}}(G_\Sigma^*, J)} \exp\left(\sum_{e^* = u^* v^* \in E_\Sigma^*} J_{e^*} \sigma_{u^*} \sigma_{v^*} \right),$$

where $Z_{\mathrm{Ising}}(G_\Sigma^*, J) = \sum_{\sigma \in \{-1, +1\}^{V_\Sigma^*}} \exp\left(\sum_{e^* = u^* v^* \in E_\Sigma^*} J_{e^*} \sigma_{u^*} \sigma_{v^*} \right)$ is the *Ising partition function*.

Note that to simplify notations, the inverse temperature is included in the coupling constants.
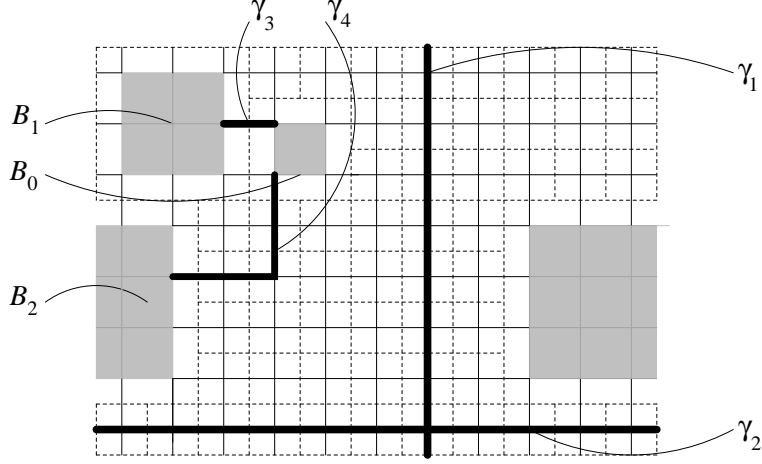
Figure 1: The graph G is a piece of $\mathbb{Z}^2$ embedded in the torus. The union of faces $(B_i)_{i \in \{0,1,2\}}$ is pictured in light grey. The graph G_Σ consists of black plain lines, and the dual graph G_Σ^* of black dotted lines. The paths $(\gamma_i)_{i=1}^4$ defining *defects* of Section 2.1 are drawn in thick black lines.

2.1 Ising model with defect lines

Let us denote $N = 2g + p - 1$. We now introduce 2^N versions of the original Ising model. Let $\gamma_1, \cdots, \gamma_N$ be N unoriented paths consisting of edges of the primal graph G_Σ, see Figure 1 for an example, where

- for $i \in \{1, \ldots, g\}$, the paths $\gamma_{2i-1}, \gamma_{2i}$ wind around the i-th handle in two transverse directions,

- for $i \in \{1, \ldots, p-1\}$, the path γ_{2g+i} joins ∂B_0 and ∂B_i.

Let ϵ be one of the 2^N possible unions of paths $\bigcup_{i \in I} \gamma_i$, where $I \subset \{1, \cdots, N\}$. Then, we change the sign of coupling constants of dual edges crossing paths of ϵ. Spin configurations are defined as above, and so is the probability measure on spin configurations. This defines the *Ising model with coupling constants (J_{e^*}) and defect condition ϵ*.

In fact, the appropriate framework for defining the Ising model with defects, is relative homology theory, see Appendices A.2, A.3 and A.4. The *first homology group of Σ relative to its boundary $\partial \Sigma$* is denoted by $H_1(\Sigma, \partial \Sigma; \mathbb{Z}/2\mathbb{Z})$. The collection of paths $(\gamma_1, \cdots, \gamma_N)$ defined above, is a representative of a basis $\Gamma = (\gamma_1, \cdots, \gamma_N)$ of the first relative homology group $H_1(\Sigma, \partial \Sigma; \mathbb{Z}/2\mathbb{Z})$.

Let ϵ denote the relative homology class of ϵ in $H_1(\Sigma, \partial \Sigma; \mathbb{Z}/2\mathbb{Z})$. Then, it will be clear from the low temperature expansion of Section 2.2 that the partition function is independent of the choice of basis and of the choice of representative of ϵ. As a consequence, we refer to this

model as the *Ising model with coupling constants (J_{e^*}) and defect condition ϵ*, and denote by $Z^{\epsilon}_{\text{Ising}}(G^*_\Sigma, J)$ the corresponding partition function. Nevertheless, since we want the change of signs of coupling constants to be well defined throughout the paper, we fix representatives of relative homology classes in $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, using the collection of paths $\gamma_1, \cdots, \gamma_N$ defined above. Note that the original Ising model introduced has defect condition $\epsilon = 0$ and ϵ is the empty collection of paths.

2.2 Low and high temperature expansion

Proposition 6 below extends the *low temperature expansion* of Kramers and Wannier [KW41a, KW41b] to the case of graphs embedded on a compact orientable surface with boundary. It consists in rewriting the Ising partition function as a sum over polygon configurations of the graph G_Σ, 'separating' clusters of ± 1 spins, see Figure 2 (left) for an example.
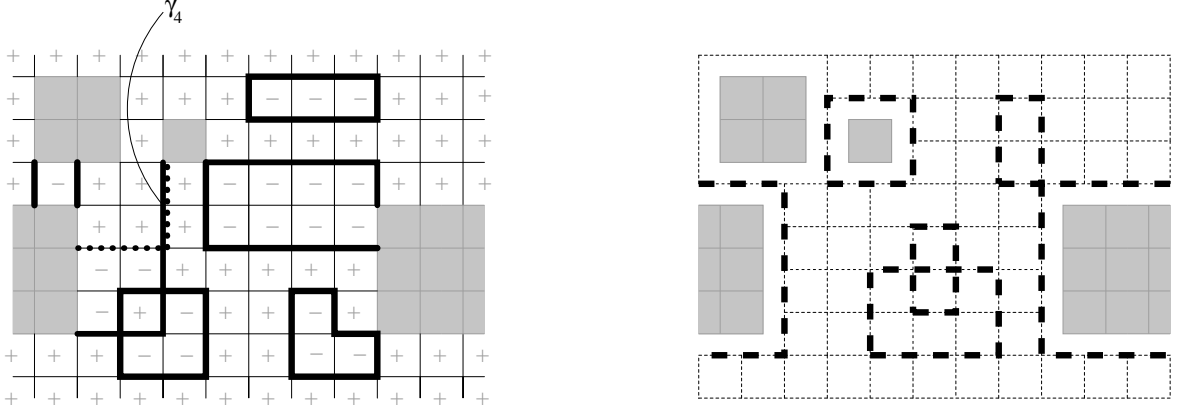

Figure 2: Left: polygon configuration of G_Σ corresponding to a spin configuration of the Ising model with defect condition $\epsilon = \gamma_4$. Right: polygon configuration of G^*_Σ.

A *polygon configuration* of G_Σ is a subset of edges of G_Σ, such that vertices not on the boundary $\partial\Sigma$ are incident to an even number of edges. There is no restriction for vertices on the boundary $\partial\Sigma$. Let us denote by $\mathcal{P}(G_\Sigma)$ the set of polygon configurations of G_Σ.

Let ϵ be a generic element of $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, and let $\mathcal{P}^\epsilon(G_\Sigma)$ denote the set of polygon configurations of G_Σ whose relative homology class in $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ is ϵ. This defines a partition of $\mathcal{P}(G_\Sigma)$:

$$\mathcal{P}(G_\Sigma) = \bigcup_{\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})} \mathcal{P}^\epsilon(G_\Sigma).$$

Proposition 6 (Low temperature expansion)**.**
For every relative homology class $\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$,

$$Z^{\epsilon}_{\text{Ising}}(G^*_\Sigma, J) = 2 \Big(\prod_{e \in E_\Sigma} e^{J_{e^*}} \Big) \sum_{P \in \mathcal{P}^\epsilon(G_\Sigma)} \prod_{e \in P} e^{-2J^*_e}. \tag{1}$$

Proof. Suppose for the moment that ϵ is the class $0 \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, so that we deal with the usual Ising model. Using the identity (2) below, one can rewrite the partition function as a statistical sum over polygon configurations separating clusters of ± 1 spins: if σ_{u^*} and σ_{v^*} are two neighboring spins of an edge $e^* = u^*v^*$, then

$$e^{J_{e^*}\sigma_{u^*}\sigma_{v^*}} = e^{J_{e^*}}\left(\delta_{\{\sigma_{u^*}=\sigma_{v^*}\}} + e^{-2J_{e^*}}\delta_{\{\sigma_{u^*}\neq\sigma_{v^*}\}}\right). \tag{2}$$

When injecting the right hand side in the expression of the Ising partition function, the product over dual edges e^* of $e^{J_{e^*}}$ can be factored out. Since primal and dual edges are in bijection, this can also be written as a product over primal edges. Then, expanding the product, we get a product of contributions for all edges separating two neighboring spins with opposite signs. These edges form a polygon configuration P^0 of G_Σ separating clusters of ± 1 spins. As a consequence P^0 has homology class 0, *i.e.* P^0 belongs to $\mathcal{P}^0(G_\Sigma)$.

Conversely, any polygon configuration of $\mathcal{P}^0(G_\Sigma)$ is the boundary of exactly two spin configurations, one obtained from the other by negating all spins, which explains the factor 2 on the right hand side of (1).

Suppose now that $\epsilon \neq 0$. In the Ising model with defect condition ϵ, coupling constants of edges crossing paths of the representative $\boldsymbol{\epsilon}$ are negated. For these edges, the relation (2) should be replaced by the following:

$$e^{-J_{e^*}\sigma_{u^*}\sigma_{v^*}} = e^{J_{e^*}}\left(\delta_{\{\sigma_{u^*}\neq\sigma_{v^*}\}} + e^{-2J_{e^*}}\delta_{\{\sigma_{u^*}=\sigma_{v^*}\}}\right).$$

Note that, when comparing to (2), the two Kronecker symbols have been exchanged. As a consequence, the construction of polygon configurations as above is slightly modified: the edge configuration, denoted by P, constructed from a spin configuration is obtained from P^0 by switching the state of every edge e in the collection of paths $\boldsymbol{\epsilon}$, see Figure 2 (left). Then, the relative homology class of P in $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ is:

$$[P] = [P^0] + [\boldsymbol{\epsilon}] = 0 + \epsilon = \epsilon.$$

As a consequence P belongs to $\mathcal{P}^\epsilon(G_\Sigma)$ and this, independently of the choice of representative of ϵ. Conversely, any element of $\mathcal{P}^\epsilon(G_\Sigma)$ is obtained twice in this way. $\square$

For the sequel, it is useful to introduce a symbol for the sum over polygon configurations of the low temperature expansion. For $\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, define

$$Z_{\text{LT}}^\epsilon(G_\Sigma, J) = \sum_{P \in \mathcal{P}^\epsilon(G_\Sigma)} \left(\prod_{e \in P} e^{-2J_{e^*}}\right).$$

The partition function of the Ising model with defect condition ϵ can thus be rewritten as:

$$Z_{\text{Ising}}^\epsilon(G_\Sigma^*, J) = 2\left(\prod_{e \in E_\Sigma} e^{J_{e^*}}\right) Z_{\text{LT}}^\epsilon(G_\Sigma, J).$$

8

Proposition 7 below extends the *high temperature expansion* [KW41a, KW41b, Wan45] to the case of graphs embedded in a compact orientable surface with boundary. It consists in rewriting the Ising partition function as a sum over polygon configurations of the graph G_Σ^*, this time. In this case, polygon configurations do not have a simple interpretation in terms of spin configurations.

A *polygon configuration* of G_Σ^* is a subset of edges such that each vertex of G_Σ^* is incident to an even number of edges, see Figure 2 (right) for an example. It is thus a union of closed cycles on G_Σ^*. Let us denote by $\mathcal{P}(G_\Sigma^*)$ the set of polygon configurations of G_Σ^*.

Let $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ be the first homology group of Σ, see Appendices A.1, A.3 and A.4. Then, to each polygon configuration of G_Σ^* is assigned its homology class in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. For every $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, we let $\mathcal{P}^\tau(G_\Sigma^*)$ denote the set of polygon configurations restricted to having homology class τ in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. This defines a partition of $\mathcal{P}(G_\Sigma^*)$:

$$\mathcal{P}(G_\Sigma^*) = \bigcup_{\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} \mathcal{P}^\tau(G_\Sigma^*).$$

Proposition 7 (High temperature expansion)**.**
For every relative homology class $\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$,

$$Z_{\text{Ising}}^\epsilon(G_\Sigma^*, J) = 2^{|V_\Sigma^*|} \Big(\prod_{e \in E_\Sigma} \cosh(J_{e^*}) \Big) \cdot \sum_{\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} \Big[(-1)^{(\tau|\epsilon)} \sum_{P^* \in \mathcal{P}^\tau(G_\Sigma^*)} \Big(\prod_{e^* \in P^*} \tanh(J_{e^*}) \Big) \Big],$$
(3)

where $(\tau|\epsilon)$ is the intersection form *evaluated at τ and ϵ: it is the parity of the number of intersections of any representative of τ and any representative of ϵ.*

For details on the intersection form, see Appendix A.5.

Proof. This result is based on yet another way of rewriting the quantity $e^{\pm J_{e^*}\sigma_{u^*}\sigma_{v^*}}$ for a dual edge $e^* = u^*v^*$ of E_Σ^*.

$$\begin{aligned} e^{\pm J_{e^*}\sigma_{u^*}\sigma_{v^*}} &= \cosh J_{e^*} \pm \sigma_{u^*}\sigma_{v^*} \sinh J_{e^*} \\ &= \cosh J_{e^*} \left(1 \pm \sigma_{u^*}\sigma_{v^*} \tanh J_{e^*} \right). \end{aligned}$$
(4)

The partition function is expanded into a sum of monomials in $(\sigma_{u^*})_{u^* \in V_\Sigma^*}$. In the expansion, the spin variables come by pairs of neighbors $\sigma_{u^*}\sigma_{v^*}$ and thus can be formally identified with the dual edge connecting u^* and v^*, associated with a weight $\pm \tanh J_{e^*}$. Each monomial is then interpreted as a subgraph of G_Σ^*, the degree of σ_{u^*} being the degree of u^* in the corresponding edge configuration. Because of the symmetry $\sigma \leftrightarrow -\sigma$, when resumming over spin configurations σ, only terms having even degree in each variable remain, giving a factor 2 per dual vertex, and other contributions cancel. As a consequence, the contributing monomials correspond to even subgraphs *i.e.* polygon configurations of $\mathcal{P}(G_\Sigma^*)$.

We now determine the sign of dual polygon configurations. Fix $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and a dual polygon configuration $P^* \in \mathcal{P}^\tau(G_\Sigma^*)$. Then, edges of P^* carrying a negative weight

are exactly those crossing paths of $\boldsymbol{\epsilon}$. As a consequence, the sign of the contribution of P^* corresponds to (-1) to the parity of the number of edges of P^* crossing paths of $\boldsymbol{\epsilon}$, this is exactly given by $(\tau|\epsilon)$. The dual polygon configuration P^* has thus sign $(-1)^{(\tau|\epsilon)}$. □

As in the case of the low temperature expansion, it is useful to introduce a notation for the sum over polygon configurations of the high temperature expansion. For $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, define:

$$Z_{\mathrm{HT}}^\tau(G_\Sigma^*, J) = \sum_{P^* \in \mathcal{P}^\tau(G_\Sigma^*)} \Big(\prod_{e^* \in P^*} \tanh(J_{e^*}) \Big).$$

The relation between (1) and (3) can then be rewritten in the following compact form. For every relative homology class $\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$:

$$Z_{\mathrm{LT}}^\epsilon(G_\Sigma, J) = 2^{|V_{\Sigma^*}|-1} \left(\prod_{e \in E_\Sigma} \frac{\cosh(J_{e^*})}{e^{J_{e^*}}} \right) \sum_{\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} \left[(-1)^{(\tau|\epsilon)} Z_{\mathrm{HT}}^\tau(G_\Sigma^*, J) \right]. \qquad (5)$$

Using the orthogonality relation

$$\sum_{\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})} (-1)^{(\tau|\epsilon)} (-1)^{(\tau'|\epsilon)} = 2^N \delta_{\tau, \tau'},$$

with $N = 2g + p - 1$, one can easily invert this relation:

$$Z_{\mathrm{HT}}^\tau(G_\Sigma^*, J) = 2^{-N-|V_{\Sigma^*}|+1} \left(\prod_{e \in E_\Sigma} \frac{e^{J_{e^*}}}{\cosh(J_{e^*})} \right) \sum_{\varepsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})} \left[(-1)^{(\tau|\epsilon)} Z_{\mathrm{LT}}(G_\Sigma, J) \right].$$

3 Double Ising model on a surface of genus g

In the whole of this section, we let G be a graph embedded in a compact, orientable surface Σ of genus g, and G^* denote its dual graph. Since Σ has no boundary, the first homology group $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ of Σ relative to its boundary, is identified with the first homology group $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

Instead of one Ising model on G^*, we now consider two copies of the Ising model, say a red one and a blue one, with the same coupling constants (J_{e^*}). These two models are not taken to be completely independent: we require that they have the same defect conditions, *i.e.* we ask that polygon configurations coming from the low temperature expansion of both spin configurations have the same homology class.

More precisely, from the point of view of the low temperature expansion, we are interested in the probability measure $\mathbb{P}_{\text{d-Ising}}$, on $\mathfrak{P} := \bigcup_{\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} \mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$, defined by, for every $(P_{\text{red}}, P_{\text{blue}}) \in \mathfrak{P}$:

$$\mathbb{P}_{\text{d-Ising}}(P_{\text{red}}, P_{\text{blue}}) = \frac{\mathcal{C} \Big(\prod_{e \in P_{\text{red}}} e^{-2J_{e^*}} \Big) \Big(\prod_{e \in P_{\text{blue}}} e^{-2J_{e^*}} \Big)}{Z_{\text{d-Ising}}(G^*, J)},$$

where $\mathcal{C} = \left(2\prod_{e\in E} e^{J_{e^*}}\right)^2$, and the partition function $Z_{\text{d-Ising}}(G^*, J)$ is given by:

$$Z_{\text{d-Ising}}(G^*, J) = \sum_{\epsilon\in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})} \sum_{(P_{\text{red}},P_{\text{blue}})\in\mathcal{P}^\epsilon(G)\times\mathcal{P}^\epsilon(G)} \mathcal{C}\left(\prod_{e\in P_{\text{red}}} e^{-2J_{e^*}}\right)\left(\prod_{e\in P_{\text{blue}}} e^{-2J_{e^*}}\right)$$
$$= \sum_{\epsilon\in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})} (Z_{\text{Ising}}^\epsilon(J))^2.$$

Given a pair $(P_{\text{red}}, P_{\text{blue}}) \in \mathfrak{P}$, and looking at the superimposition $P_{\text{red}} \cup P_{\text{blue}}$ on G, one defines two new edge configurations:

- $\text{Mono}(P_{\text{red}}, P_{\text{blue}})$: consisting of monochromatic edges of the superimposition $P_{\text{red}} \cup P_{\text{blue}}$, *i.e.* edges covered by exactly one of the polygon configuration;

- $\text{Bi}(P_{\text{red}}, P_{\text{blue}})$: consisting of bichromatic edges of the superimposition, *i.e.* edges covered by both polygon configurations.

Edges which are not in the two configurations above are covered neither by P_{blue} nor by P_{red}. In sections 3.1 and 3.2 below, we characterize these two set of edges.

3.1 Monochromatic edges

Let $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, and consider a pair of polygon configurations $(P_{\text{red}}, P_{\text{blue}})$ in $\mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$. Then, it can be realized as four pairs of Ising spin configurations $(\pm\sigma, \pm\sigma')$, each with defect type ϵ, where coupling constants are negated along the representative $\boldsymbol{\epsilon}$ of ϵ, chosen in Section 2.1.

Following Wilson [Wil11], to each of the four pairs of spin configurations, one assigns an XOR-*spin configuration* defined as follows: at every vertex, the XOR-spin is the product of the Ising-spins at that same vertex.

Note that the four pairs of spin configurations yield two distinct XOR-spin configurations, one being obtained from the other by negating all spins. As a consequence, both XOR-spin configurations have the same polygon configuration separating clusters of ± 1 spins, meaning that this polygon configuration is independent of the choice of $(\pm\sigma, \pm\sigma')$ realizing $(P_{\text{red}}, P_{\text{blue}})$, let us denote it by $\text{XOR}(P_{\text{red}}, P_{\text{blue}})$. Note also, that although the definition of σ and σ' depends on the particular choice of representative $\boldsymbol{\epsilon}$, the XOR polygon configuration does not: it is defined intrinsically from $(P_{\text{red}}, P_{\text{blue}})$.

Lemma 8. *For every pair of polygon configurations $(P_{red}, P_{blue}) \in \mathfrak{P}$, the monochromatic edge configuration $\text{Mono}(P_{red}, P_{blue})$ is exactly the XOR loop configuration $\text{XOR}(P_{red}, P_{blue})$. In particular, it is a polygon configuration of $\mathcal{P}^0(G)$.*

Proof. Fix a pair $(P_{\text{red}}, P_{\text{blue}}) \in \mathfrak{P}$ of red and blue polygon configurations of G, that is $(P_{\text{red}}, P_{\text{blue}}) \in \mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$ for some $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. Let (σ, σ') be one of the four pairs

11

of spin configurations whose low temperature expansion is $(P_{\text{red}}, P_{\text{blue}})$. We need to show that, for every edge e of G, e is monochromatic, if and only if XOR-spins at vertices u^*, v^* of the dual edge e^* are distinct.

Suppose first that e does not belong to ϵ. Then by the correspondence between contours and spins given by the low temperature expansion, the edge e is monochromatic, if and only if

$$\sigma_{u^*} \neq \sigma_{v^*} \quad \text{and} \quad \sigma'_{u^*} = \sigma'_{v^*}, \text{ when } e \text{ is only red},$$
$$\sigma_{u^*} = \sigma_{v^*} \quad \text{and} \quad \sigma'_{u^*} \neq \sigma'_{v^*}, \text{ when } e \text{ is only blue}.$$

If now e belongs to ϵ, then for the same reasons, e is monochromatic if and only if one of the two above conditions holds, with colors exchanged. In all cases, the edge e is monochromatic if and only if XOR spins at vertices u^* and v^* are distinct.

Being the boundary of some domain, the set of monochromatic edges must be a polygon configuration of G, with homology class 0. $\qquad\square$

3.2 Bichromatic edge configurations

Before describing features of bichromatic edge configurations, we recall some general facts. A polygon configuration P of the graph G separates the surface Σ into n_P connected components $\Sigma_1, \cdots, \Sigma_{n_P}$, where $n_P \geq 1$. For every $i \in \{1, \cdots, n_P\}$, Σ_i is a surface of genus g_i with boundary $\partial\Sigma_i$. The boundary is either empty or consists of cycles of Σ.

As in Section 2, G_{Σ_i} denotes the subgraph of G, whose vertex set V_{Σ_i} is $V \cap \Sigma_i$, and whose edge set E_{Σ_i} consists of edges of E joining vertices of V_{Σ_i}, from which edges on the boundary $\partial\Sigma_i$ are removed. The dual graph is denoted by $G^*_{\Sigma_i}$.

Recall that $H_1(\Sigma_i, \partial\Sigma_i; \mathbb{Z}/2\mathbb{Z})$ denotes the first homology group of Σ_i relative to its boundary. Consider the morphism Π_{Σ_i}, from $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ to $H_1(\Sigma_i, \partial\Sigma_i; \mathbb{Z}/2\mathbb{Z})$ defined as follows: for every $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, $\Pi_{\Sigma_i}(\epsilon)$ is the homology class in $H_1(\Sigma_i, \partial\Sigma_i; \mathbb{Z}/2\mathbb{Z})$ of the restriction of any representative $\boldsymbol{\epsilon}$ of ϵ to Σ_i, see Appendix A.6 for details.

The following lemma characterizes bichromatic edge configurations.

Lemma 9. *Fix $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, and let $P \in \mathcal{P}^0(G)$ be a polygon configuration, separating the surface Σ into connected components $\Sigma_1, \cdots, \Sigma_{n_P}$.*

- *If there exists a pair of polygon configurations $(P_{red}, P_{blue}) \in \mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$ such that $\text{Mono}(P_{red}, P_{blue}) = P$; then, for every $i \in \{1, \cdots, n_P\}$, the restriction of bichromatic edges to G_{Σ_i} is the low temperature expansion of an Ising configuration on $G^*_{\Sigma_i}$, with coupling constants $(2J_{e^*})$, and defect condition $\Pi_{\Sigma_i}(\epsilon)$. As a consequence, it is a polygon configuration in $\mathcal{P}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i})$.*

- *Given, for every $i \in \{1, \cdots, n_P\}$, a polygon configuration $P_i \in \mathcal{P}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i})$; then, there are $2^{n_P - 1}$ pairs $(P_{red}, P_{blue}) \in \mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$ such that $\text{Mono}(P_{red}, P_{blue}) = P$ and such that, for every $i \in \{1, \cdots, n_P\}$, the restriction of bichromatic edges to G_{Σ_i} is P_i.*

Proof.

- Suppose that there exists a pair of polygon configurations $(P_{\text{red}}, P_{\text{blue}})$ of $\mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G)$ such that $\text{Mono}(P_{\text{red}}, P_{\text{blue}}) = P$. Then, for every $i \in \{1, \cdots, n_P\}$, the restriction of bichromatic edges to G_{Σ_i} exactly consists in the restriction to G_{Σ_i} of one of two original polygon configurations. Since this polygon configuration has homology ϵ in Σ, the homology class in $H_1(\Sigma_i, \partial\Sigma_i; \mathbb{Z}/2\mathbb{Z})$ of the restriction to Σ_i is $\Pi_{\Sigma_i}(\varepsilon)$ by definition. As a consequence, the bichromatic edge configuration on Σ_i is a polygon configuration of $\mathcal{P}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i})$. Moreover, since all edges in the bichromatic configuration are present twice, and since the weight of pairs of polygon configurations is the product of the edge-weights contained in the pair of configurations, the effective weight of a bichromatic edge e is squared and becomes:

$$\left(e^{-2J_{e^*}}\right)^2 = e^{-2(2J_{e^*})},$$

which corresponds to a doubling of the coupling constants.

- There are two spin configurations, denoted by $\pm\xi$, whose low temperature expansion is P. Suppose that there exists a pair of spin configurations (σ, σ') whose low temperature expansion has P as monochromatic edges, then $\sigma\sigma' = \pm\xi$. Let us assume $\sigma\sigma' = \xi$, the argument being similar in the other case, this has the effect of adding a global factor 2 when speaking of spin configurations. The relation $\sigma\sigma' = \xi$ implies that there is freedom of choice for exactly one spin configuration, say σ, the other being determined by their product ξ.

  Consider a connected component Σ_i, and a polygon configuration $P_i \in \mathcal{P}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i})$. We want P_i to consist of doubled edges, so that in particular, it must contain all red edges. There are thus two choices for the first spin configuration of $G_{\Sigma_i}^*$, denoted by $\pm\sigma^i$. This holds for every $i \in \{1, \cdots, n_P\}$ and thus defines 2^{n_P} spin configurations $(\pm\sigma^1, \cdots, \pm\sigma^{n_P})$ of G^*. Recall that in each of the 2^{n_P} cases, the second spin configuration is determined by the condition $\sigma\sigma' = \xi$. Since on each connected component Σ_i, ξ is identically equal to ± 1, we deduce that $(\sigma')^i = \pm\sigma^i$. As a consequence, the low temperature expansion of σ' exactly consists of edges of P_i, *i.e.* P_i consists of red and blue edges. Summarizing, there are $2 \cdot 2^{n_P}$ pairs of spin configurations, or 2^{n_P-1} pairs of polygon configurations $(P_{\text{red}}, P_{\text{blue}})$, such that monochromatic edges are those of P and bichromatic edges those of P_i, $i \in \{1, \cdots, n_P\}$. Note that by construction (choice of σ^i's), each polygon configuration $P_{\text{red}}, P_{\text{blue}}$ is in $\mathcal{P}^\epsilon(G)$.

$\square$

Let $P \in \mathcal{P}^0(G)$ be a monochromatic edge configuration, and let $\epsilon \in H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. Denote by $\mathcal{W}_{\text{d-Ising}}^\epsilon[\text{Mono} = P]$ the contribution of the set

$$\{(P_{\text{red}}, P_{\text{blue}}) \in \mathcal{P}^\epsilon(G) \times \mathcal{P}^\epsilon(G) : \text{Mono}(P_{\text{red}}, P_{\text{blue}}) = P\},$$

to the partition function $(Z_{\text{Ising}}^{\epsilon}(J))^2$, and by

$$\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P] = \sum_{\epsilon \in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})} \mathcal{W}_{\text{d-Ising}}^{\epsilon}[\text{Mono} = P].$$

By the low temperature expansion of the Ising partition function, the weight of each polygon configuration P_{red}, P_{blue} is the product of edge-weights contained in the configuration. As a consequence, the contribution of $(P_{\text{red}}, P_{\text{blue}})$ can be decomposed as a product over monochromatic edges, and bichromatic edges of each of the components. Using Lemmas 8 and 9, this yields

Proposition 10. *For every $P \in \mathcal{P}^0(G)$ and every $\epsilon \in H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$,*

$$\mathcal{W}_{\text{d-Ising}}^{\epsilon}[\text{Mono} = P] = 2^{-1}\mathcal{C}\big(\prod_{e \in P} e^{-2J_{e^*}}\big)\Big(\prod_{i=1}^{n_P} 2Z_{\text{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J)\Big), \tag{6}$$

where $\mathcal{C} = \big(2\prod_{e \in E} e^{J_{e^*}}\big)^2$. Moreover, the double Ising partition function can be rewritten as:

$$Z_{\text{d-Ising}}(J) = \sum_{P \in \mathcal{P}^0(G)} \mathcal{W}_{\text{d-Ising}}[\text{Mono} = P],$$

and the probability measure $\mathbb{P}_{\text{d-Ising}}$ induces a probability measure on polygon configurations of $\mathcal{P}^0(G)$, given by:

$$\forall P \in \mathcal{P}^0(G), \quad \mathbb{P}_{\text{d-Ising}}[\text{Mono} = P] = \frac{\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P]}{Z_{\text{d-Ising}}(J)}. \tag{7}$$

4 Mixed contour expansion

In [Nie84], Nienhuis rewrites the partition function of the Ashkin-Teller model on the square lattice as a statistical sum over polygon families on G and G^* which do not intersect. We apply the same approach for the double Ising model on Σ but some care is required to keep track of the homology class of the polygon configurations involved.

We fix $\epsilon \in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$ and a polygon configuration $P \in \mathcal{P}^0(G)$. In Proposition 11, we apply the low/high temperature duality to each of the terms $Z_{\text{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J)$ involved in the expression of $\mathcal{W}_{\text{d-Ising}}^{\epsilon}[\text{Mono} = P]$ of Equation (6). This has the effect of transforming bichromatic polygon configurations of G_{Σ_i} into polygon configurations of $G_{\Sigma_i}^*$. Then, in Proposition 13, we sum over $\epsilon \in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$, and show that the outcome simplifies to a sum over polygon configurations of the dual graph, having 0 homology class in $H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$, and not intersecting P.

Proposition 11. *For every $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and every monochromatic polygon configuration $P \in \mathcal{P}^0(G)$, the contribution $\mathcal{W}_{\text{d-Ising}}^\epsilon[\text{Mono} = P]$ to $(Z_{\text{Ising}}^\epsilon(J))^2$, is equal to:*

$$\mathcal{C}'\Big(\prod_{e\in P}\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\Big)\prod_{i=1}^{n_P}\Big[\sum_{\tau^i\in H_1(\Sigma_i,\mathbb{Z}/2\mathbb{Z})}(-1)^{(\tau^i|\Pi_{\Sigma_i}(\epsilon))}\sum_{P_i^*\in\mathcal{P}^{\tau^i}(G_{\Sigma_i}^*)}\Big(\prod_{e^*\in P_i^*}\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\Big)\Big],$$

where, $\mathcal{C}' = 2^{|V^*|+1}\big(\prod_{e\in E}\cosh(2J_{e^*})\big)$.

Proof. The expression for $\mathcal{W}_{\text{d-Ising}}^\epsilon[\text{Mono} = P]$ of Equation (6) can be rewritten as:

$$\mathcal{W}_{\text{d-Ising}}^\epsilon[\text{Mono} = P] = 2^{n_P-1}\mathcal{C}\Big(\prod_{e\in P}e^{-2J_{e^*}}\Big)\Big(\prod_{i=1}^{n_P}Z_{\text{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J)\Big).$$

For every $i \in \{1, \cdots, n_P\}$, the contribution, $Z_{\text{LT}}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i}, 2J)$ is the low temperature expansion of an Ising model on vertices of $V_{\Sigma_i}^*$ with coupling constants $2J_{e^*}$ and defect condition $\Pi_{\Sigma_i}(\epsilon)$. By using the relation between Kramers and Wannier's low and high temperature expansions of (5), it can be expressed as a product

$$Z_{\text{LT}}^{\Pi_{\Sigma_i}(\epsilon)}(G_{\Sigma_i}, 2J) = \mathcal{A}_i \times \sum_{\tau^i\in H_1(\Sigma_i;\mathbb{Z}/2\mathbb{Z})}(-1)^{(\tau^i|\Pi_{\Sigma_i}(\epsilon))}Z_{\text{HT}}^{\tau^i}(G_{\Sigma_i}, 2J),$$

where

$$\mathcal{A}_i = 2^{|V_{\Sigma_i}^*|-1}\prod_{e\in E_{\Sigma_i}}\frac{\cosh(2J_{e^*})}{e^{2J_{e^*}}}.$$

Let us first compute the part which is independent of ϵ. Observing that the collection of sets of dual vertices $(V_{\Sigma_i}^*)_{i=1}^{n_P}$ is a partition of V^*, one writes:

$$2^{n_P-1}\mathcal{C}\Big(\prod_{e\in P}e^{-2J_{e^*}}\Big)\Big(\prod_{i=1}^{n_P}\mathcal{A}_i\Big) = 2^{n_P-1}2^2\Big(\prod_{e\in E}e^{2J_{e^*}}\Big)\Big(\prod_{e\in P}e^{-2J_{e^*}}\Big)2^{|V^*|-n_P}\Big(\prod_{e\in E_{\Sigma_i}}\frac{\cosh(2J_{e^*})}{e^{2J_{e^*}}}\Big).$$

Noticing that the collection of edges in the Σ_i's is exactly the set of edges of G not in P, we have:

$$2^{n_P-1}\mathcal{C}\Big(\prod_{e\in P}e^{-2J_{e^*}}\Big)\Big(\prod_{i=1}^{n_P}\mathcal{A}_i\Big) = 2^{|V^*|+1}\Big(\prod_{e\in E}e^{2J_{e^*}}\Big)\Big(\prod_{e\in P}e^{-2J_{e^*}}\Big)\Big(\prod_{e\in E\backslash P}e^{-2J_{e^*}}\cosh(2J_{e^*})\Big)$$

$$= 2^{|V^*|+1}\Big(\prod_{e\in E}\cosh(2J_{e^*})\Big)\Big(\prod_{e\in P}\cosh(2J_{e^*})^{-1}\Big).$$

Define the constant $\mathcal{C}' = 2^{|V^*|+1}\big(\prod_{e\in E}\cosh(2J_{e^*})\big)$, then by definition of $Z_{\text{HT}}^{\tau^i}(G_{\Sigma_i}, 2J)$, one deduces that the contribution of $\mathcal{W}_{\text{d-Ising}}^\epsilon[\text{Mono} = P]$ is equal to:

$$\mathcal{C}'\Big(\prod_{e\in P}\cosh(2J_{e^*})^{-1}\Big)\prod_{i=1}^{n_P}\Big[\sum_{\tau^i\in H_1(\Sigma_i,\mathbb{Z}/2\mathbb{Z})}(-1)^{(\tau^i|\Pi_{\Sigma_i}(\epsilon))}\sum_{P_i^*\in\mathcal{P}^{\tau^i}(G_{\Sigma_i}^*)}\Big(\prod_{e^*\in P_i^*}\tanh(2J_{e^*})\Big)\Big].$$

15

The proof of Proposition 11 is concluded by observing that

$$\cosh(2J_{e^*})^{-1} = \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}, \quad \text{and} \quad \tanh(2J_{e^*}) = \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}.$$

□

In order to have an explicit expression for the contribution of P to the partition function $Z_{\text{d-Ising}}(J)$, we need to sum the quantities $\mathcal{W}^{\epsilon}_{\text{d-Ising}}[\text{Mono} = P]$ of Proposition 11 over $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. This is the object of the next proposition.

Proposition 12. *For every polygon configuration $P \in \mathcal{P}^0(G)$, the contribution of P to $Z_{\text{d-Ising}}(J)$ is:*

$$\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P] = \mathcal{C}_{\text{I}} \Big(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}} \Big) \sum_{\{P^* \in \mathcal{P}^0(G^*): \, P \cap P^* = \emptyset\}} \Big(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}} \Big).$$

where $\mathcal{C}_{\text{I}} = 2^{|V^*| + 2g + 1} \left(\prod_{e \in E} \cosh(2J_{e^*}) \right)$.

Proof. To simplify notations, let us write the product of weights of edges in polygon configurations as follows:

$$\Theta(P) = \prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}, \quad \Theta^*(P_i^*) = \prod_{e^* \in P_i^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}, \quad \text{for } i \in \{1, \cdots, n_P\}.$$

The contribution $\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P]$ can be rewritten as:

$$\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P] = \mathcal{C}' \Theta(P) \sum_{\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} \prod_{i=1}^{n_P} \Big[\sum_{\tau^i \in H_1(\Sigma_i, \mathbb{Z}/2\mathbb{Z})} (-1)^{(\tau^i | \Pi_{\Sigma_i}(\epsilon))} \sum_{P_i^* \in \mathcal{P}^{\tau^i}(G^*_{\Sigma_i})} \Theta^*(P_i^*) \Big].$$

Expanding the product over $i \in \{1, \cdots, n_P\}$ and exchanging the summation over ϵ and $(\tau^1, \cdots, \tau^{n_P})$, one obtains that $\mathcal{W}_{\text{d-Ising}}[\text{Mono} = P]$ is equal to:

$$\mathcal{C}' \Theta(P) \sum_{(\tau^1, \cdots, \tau^{n_P}) \in \prod_{i=1}^{n_P} H_1(\Sigma_i, \mathbb{Z}/2\mathbb{Z})} \prod_{i=1}^{n_P} \Big[\Big(\sum_{P_i^* \in \mathcal{P}^{\tau^i}(G^*_{\Sigma_i})} \Theta^*(P_i^*) \Big) \Big(\sum_{\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} (-1)^{\sum_{i=1}^{n_P} (\tau^i | \Pi_{\Sigma_i}(\epsilon))} \Big) \Big].$$

Let us have a closer look at the last factor: $\sum_{\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} (-1)^{\sum_{i=1}^{n_P} (\tau^i | \Pi_{\Sigma_i}(\epsilon))}$. For every $\mathcal{T} = (\tau^1, \cdots, \tau^{n_P}) \in \prod_{i=1}^{n_P} H_1(\Sigma_i, \mathbb{Z}/2\mathbb{Z})$, consider the linear form:

$$\begin{array}{rccl} \psi_{\mathcal{T}} : & H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & \epsilon & \longmapsto & \psi_{\mathcal{T}}(\epsilon) = \sum_{i=1}^{n_P} (\tau^i | \Pi_{\Sigma_i}(\epsilon)), \end{array}$$

which once evaluated on the homology class ϵ, gives the exponent of (-1) in the sum above.

Since $(-1)^{\psi_{\mathcal{T}}(\epsilon)}$ equals 1 (respectively -1) if and only if ϵ is (respectively is not) in $\ker(\psi_{\mathcal{T}})$, we can write:

$$\sum_{\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} (-1)^{\psi_{\mathcal{T}}(\epsilon)} = \#\{\epsilon \ : \ \psi_{\mathcal{T}}(\epsilon) = 0\} - \#\{\epsilon \ : \ \psi_{\mathcal{T}}(\epsilon) = 1\}$$
$$= 2\#\ker(\psi_{\mathcal{T}}) - \#H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}). \tag{8}$$

If $\psi_{\mathcal{T}}$ is a non constant form, its kernel is a hyperplane of $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, thus having dimension $2g - 1$ and cardinality 2^{2g-1}, implying that the sum over ϵ of (8) is equal to $2.2^{2g-1} - 2^{2g} = 0$. Therefore, the contribution of $\mathcal{T}$ is non zero if and only if $\psi_{\mathcal{T}}$ is identically zero, and in this case, the sum over ϵ of (8) gives a factor:

$$2\#H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) - \#H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) = \#H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) = 2^{2g}.$$

Let us now determine which are the $\mathcal{T}$'s of $(\prod_{i=1}^{n_P} H_1(\Sigma_i, \mathbb{Z}/2\mathbb{Z}))$, such that $\psi_{\mathcal{T}}$ is identically zero.

First, notice that the evaluation of the intersection form $(\tau^i | \Pi_{\Sigma_i}(\epsilon))$ on $H_1(\Sigma_i; \mathbb{Z}/2\mathbb{Z}) \times H_1(\Sigma_i, \partial\Sigma_i; \mathbb{Z}/2\mathbb{Z})$ is equal to $(\pi_i(\tau^i) | \epsilon)$ on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \times H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, where π_i is the projection induced by the inclusion $\Sigma_i \subset \Sigma$, see Appendix A.6. Indeed, take a representative $\boldsymbol{\tau}^i$ of τ^i in Σ_i. Counting intersections with the restriction of a representative $\boldsymbol{\epsilon}$ of ϵ to Σ_i is the same as counting intersections with the whole $\boldsymbol{\epsilon}$, since $\boldsymbol{\tau}^i$ is confined to Σ_i and thus has no intersection with $\boldsymbol{\epsilon}$ outside of Σ_i. Therefore, $\psi_{\mathcal{T}}(\epsilon)$ can be rewritten as:

$$\psi_{\mathcal{T}}(\epsilon) = \sum_{i=1}^{n_P} (\tau^i | \Pi_{\Sigma_i}(\epsilon)) = \Big(\sum_{i=1}^{n_P} \pi_i(\tau^i) \Big| \epsilon\Big).$$

Since the intersection form is non degenerate on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, $\psi_{\mathcal{T}}$ is zero if and only if $\sum_{i=1}^{n_P} \pi_i(\tau^i) = 0 \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, $i.e.$ the homology class on the surface Σ of the whole polygon configuration $P^* = P_1^* \cup \cdots \cup P_{n_P}^*$ is zero, $i.e$, $P^* \in \mathcal{P}^0(G^*)$. $\square$

As an interesting corollary, we obtain a mixed contour expansion for the partition function of the double Ising model, and the corresponding expression for the double Ising probability measure of monochromatic polygon configurations which, by Lemma 8, are the XOR polygon configurations, .

Corollary 13.

- *The double Ising partition function can be rewritten as:*

$$Z_{\text{d-Ising}}(G^*, J) = \mathcal{C}_{\text{I}} \sum_{\{(P, P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*): \, P \cap P^* = \emptyset\}} \Big(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big)\Big(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big),$$

where $\mathcal{C}_{\text{I}} = 2^{|V^*|+2g+1} \left(\prod_{e \in E} \cosh(2J_{e^*})\right)$.

- *For every polygon configuration $P \in \mathcal{P}^0(G)$:*

$$\mathbb{P}_{\text{d-Ising}}[\text{XOR} = P] = \frac{\left(\prod_{e \in P} \dfrac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}\right) \displaystyle\sum_{\{P^* \in \mathcal{P}^0(G^*):\, P \cap P^* = \emptyset\}} \left(\prod_{e^* \in P^*} \dfrac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}\right)}{\displaystyle\sum_{\{(P,P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*):\, P \cap P^* = \emptyset\}} \left(\prod_{e \in P} \dfrac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}\right)\left(\prod_{e^* \in P^*} \dfrac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}\right)}.$$

Note that one can see Kramers and Wannier's duality on this expression: the duality relation between coupling constant

$$\tanh J^* = e^{-2J}$$

exchanges the expression for an edge of P and a dual edge of P^*.

5 Quadri-tilings and polygon configurations

In the whole of this section, we let G be a graph embedded in a compact, orientable surface Σ of genus g, and G^* denote its dual graph. For the moment, we forget about the double Ising model. The goal of this section is to explicitly construct pairs of non-intersecting polygon configurations of G and G^*, from a dimer model on a decorated, bipartite version G^Q of G, called *quadri-tilings* [dT07a].

This construction is done in two steps. The first step uses a mapping of Nienhuis [Nie84], which constructs pairs of non-intersecting primal and dual polygon configurations, from 6-vertex configurations of the medial graph; this is the subject of Section 5.1. The second step consists in using Wu-Lin/Dubédat's mapping [WL75, Dub11b] from the 6-vertex model of the medial graph to the bipartite dimer model on the decorated graph G^Q. This is the subject of Section 5.2.

Using the above results and those of Section 4, Theorem 18 proves that XOR loops of the double Ising model have the same law as primal polygon configurations of the bipartite dimer model.

5.1 6-vertex model and polygon configurations

The *medial graph* G^M of the graph G is defined as follows. Vertices of G^M correspond to edges of G. Two vertices of the medial graph are joined by an edge if the corresponding edges in the primal graph are incident. Observe that G^M is also the medial graph of the dual graph G^*, and that vertices of the medial graph all have degree four. Figure 3 represents the medial graph of a subset of $\mathbb{Z}^2$.

A *6V-configuration* or an *ice-type configuration* is an orientation of edges of G^M, such that every vertex has exactly two incoming edges [Lie67]. An equivalent way of defining 6-vertex configurations uses edge configurations instead of orientations, as represented in Figure 4.
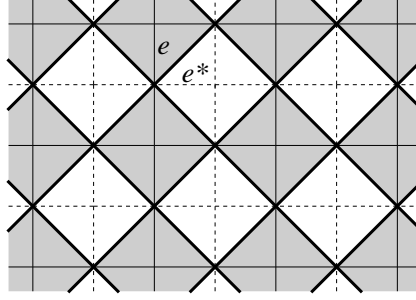
Figure 3: The medial graph of $\mathbb{Z}^2$: plain lines represent $\mathbb{Z}^2$, dotted lines represent the dual graph $\mathbb{Z}^2$, and thick plain lines represent the medial graph $(\mathbb{Z}^2)^M$. Grey (resp. white) faces of the medial graph correspond to primal (resp. dual) vertices of the initial graph.

This approach is more useful in our context, so that we define a 6-*vertex configuration* to be an edge configuration, such that around every vertex of G^M, there is an even number of consecutive present edges.
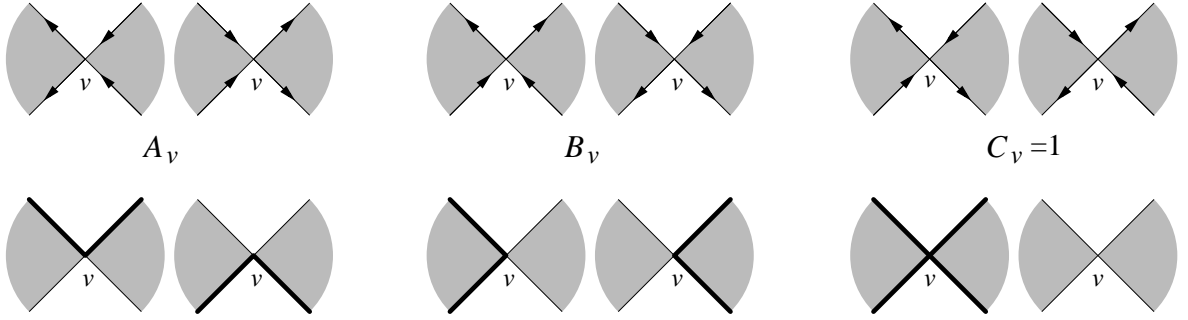

Figure 4: The six possible local configurations around a vertex v of the 6-vertex model, and their respective weights: arrow representation (top), and the even degree subgraph representation (bottom).

In order to make the 6-vertex model a model of statistical mechanics, weights are associated to local configurations around a vertex, and the probability of a 6-vertex configuration is taken to be proportional to the product of its local weights. In absence of external field, the weights of complementary local configurations are taken to be equal: there are thus three parameters for each vertex v of the medial graph G^M, denoted by A_v, B_v and C_v. Since, multiplying these three parameters by the same positive constant does not change the measure, we set $C_v = 1$, see also Figure 4. Let us denote by $Z_{6\text{-vertex}}(G^M, (A, B))$ the partition function of this model.

Mapping I [Nie84]. Consider the following combinatorial mapping from 6-vertex configurations to edge configurations of the primal and dual graph: whenever a vertex of G^M has

two neighboring edges in the 6 vertex configuration, put the edge of G or G^* separating the present and the absent edges, see Figure 5. The following lemma characterizes this mapping, see also [Nie84].
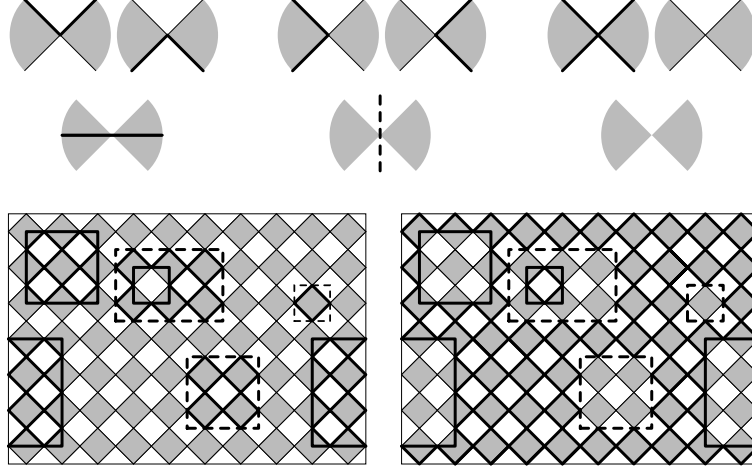

Figure 5: Mapping on the local level (top), and on the global level (bottom).

Lemma 14.

- *Mapping I associates to a 6-vertex configuration a pair of polygon configurations (P, P^*), which do not intersect and such that the homology class of $P \cup P^*$ in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is 0.*

- *Given a pair of polygon configurations (P, P^*) as above, there are exactly two 6-vertex configurations which are mapped to (P, P^*).*

Proof. A 6-vertex configuration consists of clusters of present/absent edges, see Figure 5. The primal/dual edge configuration assigned by the mapping consists of the boundary of those clusters. It is thus a polygon configuration with 0 homology class. A primal and the corresponding dual edge cannot intersect, since that would correspond to a configuration on G^{M} where around a vertex there is alternatively one edge present, one edge absent, then again one present, one absent, which is a forbidden local configuration for the 6-vertex model.

Conversely suppose we are given a pair of polygon configurations (P, P^*) as above. The fact that the homology class of $P \cup P^*$ in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is 0, exactly means that it is the boundary of domains that can be painted alternatively in two colors consistently. Put all edges of G^{M} in domains of one color, and remove all edges in domains of the other color. In this way, one exactly obtains two valid configurations of the 6-vertex model, one being the complement of the other, depending on which color is used to represent present edges, see also Figure 5. $\square$

Let us denote by $\mathcal{P}^0(G \cup G^*)$ the set of pairs (P, P^*) of polygon configurations of G and G^* respectively, such that the union $P \cup P^*$ has 0 homology class in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.

Recall that to every edge e of the primal graph (resp. e^* of the dual graph), corresponds a vertex v of the medial graph, which we denote by $v(e)$ (resp. $v(e^*)$). Using this fact, one can naturally define a weight function on edges of G and G^*:

$$\forall e \in G, \quad a_e := A_{v(e)}; \quad \forall e^* \in G^*, \quad b_{e^*} := B_{v(e^*)}.$$

With this choice of weights and using Lemma 14, we obtain the following.

Lemma 15.

$$Z_{6\text{-vertex}}(G^{\mathrm{M}}, (A, B)) = 2 \sum_{\{(P, P^*) \in \mathcal{P}^0(G \cup G^*):\, P \cap P^* = \emptyset\}} \prod_{e \in P} a_e \prod_{e^* \in P^*} b_{e^*}.$$

5.2 Quadri-tilings and 6-vertex model

Let us define yet another graph built from the graph G. The *quadri-tiling* graph of G, denoted by G^{Q}, is the decorated graph obtained from G^{M} by replacing every vertex by a decoration which is a quadrangle. The graph G^{Q} is bipartite and can be drawn on the same surface as G. Edges shared by G^{Q} and G^{M} are referred to as *external* edges, and those inside the decorations as *internal*.
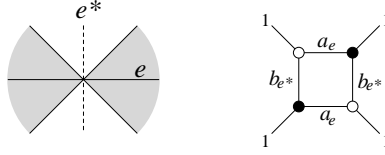

Figure 6: Decoration of a vertex of the medial graph G^{M} (left) to obtain a piece of the quadri-tiling graph G^{Q} (right).

A *dimer configuration* of G^{Q}, also known as a *perfect matching* is a spanning subgraph of G^{Q} where every vertex has degree exactly one. Let us denote by $\mathcal{M}(G^{\mathrm{Q}})$ the set of dimer configurations of the graph G^{Q}. In a particular decoration of G^{Q}, a dimer configuration of G looks like one of the seven possibilities represented in Figure 7 (top).

Assigning positive weights $(w_e)_{e \in E^{\mathrm{Q}}}$ to edges of G^{Q}, the *dimer Boltzmann measure*, denoted $\mathbb{P}_{\mathrm{Q}}$, is defined by:

$$\forall M \in \mathcal{M}(G^{\mathrm{Q}}), \quad \mathbb{P}_{\mathrm{Q}}(M) = \frac{\prod_{e \in M} w_e}{Z_{\mathrm{dimer}}(G^{\mathrm{Q}}, w)},$$

where $Z_{\mathrm{dimer}}(G^{\mathrm{Q}}, w) = \sum_{M \in \mathcal{M}(G^{\mathrm{Q}})} \prod_{e \in M} w_e$ is the dimer partition function. This defines a model of statistical mechanics, called the *dimer model* on G^{Q}.

Mapping II [WL75, Dub11b]. Requiring exterior edges to match yields a mapping from dimer configurations of G^{Q} to 6-vertex configurations of G^{M}, see Figure 7. This mapping between local configurations is one to one except in the empty edge case where this mapping is two-to-one.
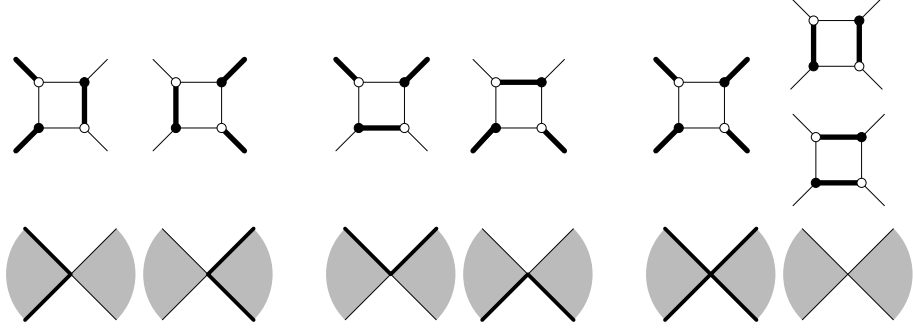
Figure 7: The local configurations of a quadri-tiling and the corresponding local 6-vertex configurations.

We now choose weights of edges in a specific way. Let A and B be positive functions on vertices of the medial graph G^{M}, defining weights of local 6-vertex configuration; and let a and b be the induced weight functions on edges of G and G^* respectively. The weight function $w^{a,b}$ on edges of G^{Q} is defined as follows, see also Figure 6:

$$w_e^{a,b} = \begin{cases} 1 & \text{if } e \text{ is an external edge} \\ a_{\bar{e}} & \text{if } e \text{ is an interior edge, parallel to a primal edge } \bar{e} \\ b_{\bar{e}^*} & \text{if } e \text{ is an interior edge, parallel to a dual edge } \bar{e}^*. \end{cases}$$

Let us denote by $Z_{\mathrm{dimer}}(G^{\mathrm{Q}}, (A, B))$ the corresponding partition function. From now on, we suppose that local weights of the 6-vertex model satisfy the relation:

$$\forall v \in V(G^{\mathrm{M}}), \quad A_v^2 + B_v^2 = 1.$$

This implies that, $\forall v \in V(G^{\mathrm{M}})$, $\Delta_v = \frac{A_v^2 + B_v^2 - C_v^2}{2 A_v B_v} = 0$, *i.e.* the model is *free fermionic*.

Using the weight functions a and b, the free fermionic condition can be rewritten as:

$$\forall e \in E, \quad a_e^2 + b_{e^*}^2 = 1. \tag{9}$$

With this choice of weight, we thus obtain the following.

Lemma 16. *When local weights of the 6-vertex model satisfy the free fermionic relation, Mapping II is weight preserving and:*

$$Z_{\mathrm{dimer}}(G^{\mathrm{Q}}, (A, B)) = Z_{6\text{-vertex}}(G^{\mathrm{M}}, (A, B)).$$

Let us now choose weights of edges of G^{Q} to depend on coupling constants (J_{e^*}) of the double Ising model:

$$\forall e \in E, \ a_e = \frac{2 e^{-2 J_{e^*}}}{1 + e^{-4 J_{e^*}}}, \ \text{ and } \ \forall e^* \in E^*, \ b_{e^*} = \frac{1 - e^{-4 J_{e^*}}}{1 + e^{-4 J_{e^*}}}. \tag{10}$$

Then, it is straightforward to check that a and b satisfy the free fermionic condition (9):

$$\forall e \in E, \ a_e^2 + b_{e^*}^2 = \frac{4e^{-4J_{e^*}} + (1 - e^{-4J_{e^*}})^2}{(1 + e^{-4J_{e^*}})^2} = 1.$$

It should be noted that in the more general case of the Ashkin-Teller model, when spins of the two Ising models interact, the mapping with the 6-vertex model still holds [Nie84, Sal87], but the model is not a free fermion any more.

Since weights a and b depend on coupling constants (J_{e^*}), we set J as argument for the corresponding dimer model partition function.

Recall that the mixed contour expansion of the double Ising partition function, see Corollary 13, involves pairs of non intersecting primal and dual polygon configurations, each of which has 0 homology class in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. We want to take the same restriction here.

Consider a dimer configuration M of the graph G^{Q}, then Mapping II assigns to M a 6-vertex configuration, and Mapping I assigns to this 6-vertex configuration a pair of non-intersecting polygon configuration of $\mathcal{P}^0(G \cup G^*)$. Let us denote this pair by $\mathrm{Poly}(M) = (\mathrm{Poly}_1(M), \mathrm{Poly}_2(M))$.

We restrict ourselves to dimer configurations M such that $\mathrm{Poly}_1(M)$ has 0 homology class, and denote by $\mathcal{M}^0(G^{\mathrm{Q}})$ this set. Note that since the superimposition $\mathrm{Poly}_1(M) \cup \mathrm{Poly}_2(M)$ has 0 homology class, this automatically implies that $\mathrm{Poly}_2(M)$ also has 0 homology. Let $\mathbb{P}^0_{\mathrm{Q}}$ be the corresponding dimer Boltzmann-measure, and $Z^0_{\mathrm{dimer}}(G^{\mathrm{Q}}, J)$ be the corresponding partition function.

Let $(P, P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*)$ such that $P \cap P^* = \emptyset$. Denote by $\mathcal{W}_{\mathrm{Q}}[\mathrm{Poly} = (P, P^*)]$, the contribution of the set:

$$\{M \in \mathcal{M}(G^{\mathrm{Q}}) : \ \mathrm{Poly}(M) = (P, P^*)\} \subset \mathcal{M}^0(G),$$

to the partition function $Z^0_{\mathrm{dimer}}(G^{\mathrm{Q}}, J)$. Then, as a consequence of Lemmas 15 and 16, we have:

Proposition 17. *When weights assigned to edges of the graph G^{Q} are chosen as in Equation (10), we have for all $(P, P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*)$, such that $P \cap P^* = \emptyset$:*

$$\mathcal{W}_{\mathrm{Q}}[\mathrm{Poly} = (P, P^*)] = 2\Big(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big)\Big(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big),$$

Moreover, the dimer model partition function can be written as:

$$Z^0_{\mathrm{dimer}}(G^{\mathrm{Q}}, J) = 2 \sum_{\{(P,P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*): \ P \cap P^* = \emptyset\}} \Big(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big)\Big(\prod_{e^* \in P^*} \frac{1 - e^{-4J_{e^*}}}{1 + e^{-4J_{e^*}}}\Big)$$

and the probability measure $\mathbb{P}^0_Q$ induces a probability measure on polygon configurations of $\mathcal{P}^0(G)$, given by:

$$\mathbb{P}^0_Q[\text{Poly}_1 = P] = \frac{\left(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right) \sum\limits_{\{P^* \in \mathcal{P}^0(G^*): P \cap P^* = \emptyset\}} \left(\prod_{e^* \in P^*} \frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)}{\sum\limits_{\{(P,P^*) \in \mathcal{P}^0(G) \times \mathcal{P}^0(G^*): P \cap P^* = \emptyset\}} \left(\prod_{e \in P} \frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right)\left(\prod_{e^* \in P^*} \frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)}.$$

Combining Corollary 13 and Proposition 17 yields the following.

Theorem 18.

- *The double Ising partition function and the dimer model partition function are equal up to an explicit constant:*

$$Z_{\text{d-Ising}}(G^*, J) = 2^{|V^*|+2g}\left(\prod_{e \in E} \cosh(2J_{e^*})\right) Z^0_{\text{dimer}}(G^Q, J).$$

- *XOR-polygon configurations of the double Ising model on G^* have the same law as Poly_1 configurations of the corresponding dimer model on the bipartite graph G^Q:*

$$\forall P \in \mathcal{P}^0(G), \ \mathbb{P}_{\text{d-Ising}}[\text{XOR} = P] = \mathbb{P}^0_Q[\text{Poly}_1 = P].$$

Note that the first part of Theorem 18 can also be deduced from the results of [Dub11b] and [CD12].

Since the quadri-tiling model is a bipartite dimer model, it can be studied in great details the tools of Kasteleyn theory, which we recall some elements of in the next section. These tools can thus be used to study the distribution of the XOR Ising configurations.

5.3 Kasteleyn theory

We now recall some elements of the Kasteleyn theory for bipartite dimer models and apply it to the dimer model on G^Q. This simplified version for bipartite graphs of the more general theory developed by Kasteleyn [Kas67] is due to Percus [Per69]. The main tool is the *Kasteleyn matrix*, defined as follows in the bipartite case:

- rows (resp. columns) are indexed by white (resp. black) vertices;

- the absolute value of an entry is 0 if the corresponding white and black vertices are not adjacent, and is the dimer weight of the edge formed by these vertices when they are adjacent;

- signs of the entries are chosen in such a way that around all (bounded) faces, the number of minus signs around a face has the same parity as half of the degree of the face, minus 1.

If the graph G^Q is planar, then the partition function of the model is, up to a global sign, the determinant of the Kasteleyn matrix. Indeed, when expanding the determinant of K as a sum over permutations, the only non zero terms are those corresponding to dimer configurations, and their absolute value is the correct weight. The third condition about signs is here to compensate the signatures of the permutations, so that all the terms exactly have the same sign. Kasteleyn showed [Kas61, Kas67] that such a choice of signs exists (and is essentially unique): changing the signs of all edges around a particular vertex still yields a choice of signs satisfying the third condition, and one can pass from one valid choice to another by a succession of such operations. In order to get the right global sign, one can chose a reference dimer configuration M_0, giving a bijection between white and black vertices, agree that the order chosen for rows and columns of K is compatible with this bijection, and chose signs so that all entries of K corresponding to dimers of the reference configuration have sign $+$.

When superimposing a dimer configuration M of G^Q with the reference dimer configuration M_0, each vertex of G^Q is incident to exactly two edges of the superimposition, so that we obtain a family of non-intersecting loops and doubled edges covering all vertices of G^Q.

If the graph G^Q is embedded in a surface Σ of genus $g > 0$, loops of the superimposition also live on Σ and may have non-trivial homology in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. For the purpose of this paper, it is convenient to fix as reference dimer configuration M_0 the one covering all internal edges of decorations parallel to dual edges of G^*. Then we have the following useful lemma.

Lemma 19. *Let M be a dimer configuration of G^Q. Then, the homology classes of $M_0 \cup M$ and $\mathrm{Poly}_1(M)$ in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ are equal.*

Proof. In order to prove that the homology classes are the same, it is sufficient to prove that they give the same result when computing the intersection form against any homology class $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.

Let us fix such a class $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. Let $\boldsymbol{\tau}$ be a representative of the class τ, realized as a path on G^*. We now show that the parity of the number of intersections between $\mathrm{Poly}_1(M)$ and $\boldsymbol{\tau}$, is equal to the number of intersection between $M \cup M_0$ and $\boldsymbol{\tau}$. All intersections occur in the interior of dual edge used by $\boldsymbol{\tau}$.

Fix e^* a dual edge used by $\boldsymbol{\tau}$. From the mappings above, the edge e belongs to $\mathrm{Poly}_1(M)$ if and only if the number of interior edges parallel to e in the corresponding rhombus covered by dimers in M is odd (see Figures 5 and 7). Since edges of M_0 are parallel to e^*, the parity of the number of intersections with e^* will be the same for $\mathrm{Poly}_1(M)$ as for $M \cup M_0$. Since this holds for every dual edge belonging to $\boldsymbol{\tau}$, it holds for $\boldsymbol{\tau}$. Therefore, the homology classes for $\mathrm{Poly}_1(M)$ and $M \cup M_0$ are the same.

$\square$

Kasteleyn theory of graphs embedded in surface of genus $g > 0$ is more complicated than for planar graphs. There is a topological obstruction for the existence of a sign distribution on edges giving every dimer configuration a $+$ sign in the determinant expansion of a Kasteleyn

matrix. There still exist choices of signs satisfying the third condition for a Kasteleyn matrix, but there is not just one, as in the planar case, but 2^{2g} classes of choices of signs. Therefore we can define 2^{2g} non-equivalent Kasteleyn matrices.

From one of them, denoted by $K^{(0)}$, one can construct the others non-equivalent matrices denoted by $K^{(\epsilon)}$, $\epsilon \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, by multiplying the sign of all edges crossing a representative of ϵ by -1.

In the expansion of the determinant of each of the 2^{2g} matrices, there are terms with different signs: The sign of a dimer configuration M in the expansion depends on ϵ and the homology class in $\mathbb{Z}/2\mathbb{Z}$ of the loops in the superimposition of M and of the reference dimer configuration M_0. With our choice of reference dimer configuration, it thus depends on the homology class of primal contours of $\text{Poly}_1(M)$. The determinant of each Kasteleyn matrix has thus the following form

$$\det K^{(\epsilon)} = \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} s_{\alpha, \epsilon} Z^{(\alpha)}_{\text{dimer}},$$

where $Z^{(\alpha)}_{\text{dimer}}$ is the partition function for dimer configurations such that Poly_1 has homology class α and $s_{\alpha, \epsilon}$ is a sign depending only on ϵ and α.

It turns out that the linear relations between $\det K^{(\epsilon)}$ and $Z^{(\alpha)}_{\text{dimer}}$ can be inverted explicitly and each $Z^{(\alpha)}_{\text{dimer}}$ is a linear combination of the 2^{2g} determinants of Kasteleyn matrices [DZM$^+$96, GL99, Tes00, CR07, CR08]. This is in particular true for $Z^{(0)}_{\text{dimer}}$.

Classically, when studying dimer models one is interested in the full partition function

$$Z_{\text{dimer}} = \sum_{\alpha \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})} Z^{(\alpha)}_{\text{dimer}},$$

which is also a linear combination of the determinants of Kasteleyn matrices.

Many properties of the dimer model with the full partition function are deduced from the study of Kasteleyn matrices. As a consequence, most of them can be readily obtained also for the restricted model where the homology class of configurations is fixed, since it is just a matter of considering another linear combination of determinants.

This discussion of the relation of signs and homology considerations and the selection of particular homology classes via adapted linear combinations is not specific to the quadritiling model, but applies as is to any bipartite dimer. It applies in fact to any dimer model, where the more general Kasteleyn theory applies, using *full* Kasteleyn matrices with rows and columns indexed by *all* vertices of the graph, and Pfaffians instead of determinants. We refer the reader to [CR07] for an intrinsic geometric interpretation of coefficients in the linear combination in this general context. The low temperature polygon configurations of the Ising model can be mapped via Fisher's correspondence [Fis66] to a non-bipartite dimer model. Restricting the homology class of the polygon configurations can also be obtained on the dimer side with an appropriate linear combination of Pfaffians of Kasteleyn matrices.

6 The double Ising model at criticality on the whole plane

After having discussed in much generality the case of finite surfaces of genus g, we now want to consider the case of infinite planar graphs. From now on, we restrict ourselves to a special kind of graphs, the so-called *isoradial graphs*, with specific values of the coupling constants for the Ising model.

6.1 Isoradial graphs

Definition 6.1. An *isoradial graph* [Ken02, KS05] is a planar graph with a proper embedding having the property that every face is inscribed in a circle of fixed radius, which can be taken equal to 1.

The regular square, triangular and hexagonal lattices with their standard embedding are isoradial. A fancier example is given in Figure 8 (left).
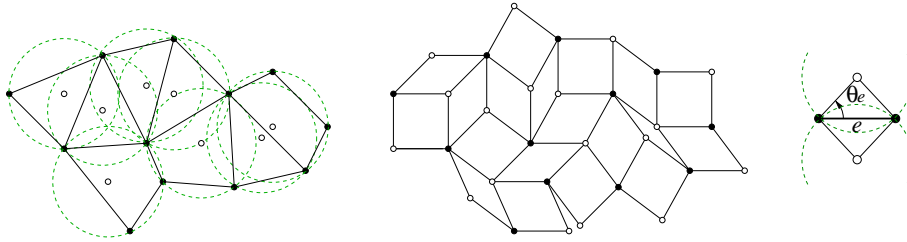

Figure 8: Left: an example of isoradial graph. Middle: the corresponding rhombus graph. Right: the half-rhombus angle θ_e associated to an edge e.

The center of the circumscribing circle of a face can be identified with the corresponding dual vertex, implying that the dual of an isoradial graph is also isoradial.

The dual of the medial graph G^{M}, called the *diamond graph* of G and denoted by $G^\diamond$, has as set of vertices the union of those of G and G^*. There is an edge between $v \in V$ and $f^* \in V^*$ if and only if v is on the boundary of the face f corresponding to the dual vertex f^*, see Figure 8 (middle) for an example. Faces of $G^\diamond$ are rhombi with edge length 1, diagonals of which correspond to an edge of G and its dual edge. To each edge e of G we can therefore associate a geometric angle $\theta_e \in (0, \pi/2)$, which is the half-angle of the rhombus containing e, measured between e and the edge of the rhombus, see Figure 8 (right). The family (θ_e) encodes the geometry of the embedding of the isoradial graph.

Note that in general, an isoradial graph has many isoradial embeddings, each of which gives a new collection of angles.

6.2 Statistical mechanics on isoradial graphs

When defining a statistical mechanical model on an isoradial graph, it is natural to relate statistical weights to the geometry of the embedding, and thus to chose the parameters attached to an edge (coupling constants for Ising, probability to be open for percolation, weight of an edge for dimer models, conductances for spanning trees or random walk,...) to be functions of the half-angle of that edge. For the Ising model, discrete integrability considerations (invariance under star-triangle transformations) and a self-duality argument yields the following expression for the interacting constants:

$$
J_e = J(\theta_e) = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right).
$$

This expression for J_e for $\theta_e = \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{6}$ coincides with the critical value of the square, hexagonal and triangular lattices respectively. The Ising model with these coupling constants and is known to exhibit a critical behavior [BDT10, BDT11, CS09]. We therefore refer to these values as *critical coupling constants*.

We suppose that the isoradial graph G is infinite, in the sense that the union of all rhombi of the diamond graph of G covers the whole plane.

Consider the corresponding bipartite dimer model on the infinite decorated quadri-tiling graph G^Q. Then, the correspondence for the weights described in Section 5.2, Equation (10) yields in this particular context:

$$
\forall e \in E, \ a_e = a(\theta_e) = \cos \theta_e, \quad \forall e^* \in E^*, \ b_{e^*} = b(\theta_{e^*}) = \cos \theta_{e^*} = \sin \theta_e.
$$

Notice that with a particular embedding of the decoration, namely when external edges have length 0, and a and b edges form rectangles joining the mid-points of edges of each rhombus, the quadri-tiling graph G^Q is itself an isoradial graph with rhombi of edge length $\frac{1}{2}$, and that weights (up to a global multiplicative factor of $\frac{1}{2}$) are those introduced by Keynon [Ken02] to define critical dimer models on isoradial graphs.

It occurs that for the Ising and dimer models on infinite isoradial graphs with critical weights indicated above, it is possible to construct Gibbs probability measures [dT07a, BDT11], extending the Boltzmann probability measures in the DLR sense: conditional on the configuration of the model outside a given bounded region of the graph, the probability measure of a configuration inside the region is given by the Boltzmann probability measure defined by the weights above (and the proper boundary conditions). These measures have the wonderful property of *locality*: the probability of a local event only depends on the geometry of a neighborhood of the region where the event takes place. This means in particular that changing the isoradial graph outside of this region does not affect the probability.

We can therefore consider the critical Ising model (resp. dimer, in particular quadri-tilings models) on a general infinite isoradial graph, as being that particular Gibbs probability measures on Ising configurations (resp. dimers configurations) of that infinite graph.

We now work with a fixed infinite isoradial graph G. We denote by $\mathbb{P}^{\infty}_{\text{Ising}}$ the measure on configurations of the critical Ising model on G^*. By taking two independent copies of the critical Ising model on G^*, we get the Gibbs measure for the critical double Ising model $\mathbb{P}^{\infty}_{\text{d-Ising}} = \mathbb{P}^{\infty}_{\text{Ising}} \otimes \mathbb{P}^{\infty}_{\text{Ising}}$, from which XOR contours can be constructed, as in the finite case. We denote by $\mathbb{P}^{\infty}_{\text{Q}}$ the Gibbs measure on dimer configurations of the infinite graph G^{Q}.

6.3   Loops of the critical XOR Ising model on isoradial graphs

It turns out that the identity in law between polygon configurations of the critical XOR Ising model on G^* and those of the corresponding bipartite dimer model on G^{Q} remains true in the context of infinite isoradial graphs at criticality:

Theorem 20. *Let G be an infinite isoradial graph. The measure induced on polygon configurations of the critical XOR Ising model on G^*, and the measure induced on* primal contours *of the corresponding critical bipartite dimer model on G^{Q} have the same law: for any finite subset of edges $\mathcal{E} = \{e_1, \ldots, e_n\}$,*

$$\mathbb{P}^{\infty}_{\text{d-Ising}}[\mathcal{E} \subset \text{XOR}] = \mathbb{P}^{\infty}_{\text{Q}}[\mathcal{E} \subset \text{Poly}_1]. \tag{11}$$

Proof. We first prove (11) when G is a bi-periodic isoradial graphs. In that case, the quotient of G by the translation lattice Λ can be viewed as an isoradial graph on the torus. Quotienting G by sublattices $n\Lambda$, with $n \in \mathbb{N}^+$ gives graphs G_n on larger and larger tori, as n increases.

If we apply Theorem 18 to the specific case of the double Ising model on a toroidal, isoradial graph G_n^* with critical coupling constants, we know that on G_n, XOR polygon configurations have the same law as primal polygon configurations of the corresponding bipartite dimer model on G_n^{Q}.

On the one hand, the graph G^{Q} being bipartite, we know [KOS06] that the sequence of Boltzmann measures $\mathbb{P}^{G_n}_{\text{Q}}$ of the dimer model on G_n^{Q}, without any restriction on the homology of (primal) cycles, converges to the Gibbs measure $\mathbb{P}^{\infty}_{\text{Q}}$ for quadri-tilings on G^{Q}, studied in [dT07a]. The same arguments can be used to show that the measure on *even* configurations of G_n^{Q} also converges to the same limit $\mathbb{P}^{\infty}_{\text{Q}}$.

On the other hand, the low temperature contours on G_n for the Ising model on G_n^* are mapped to a (non-bipartite) dimer model via Fisher's correspondence [Fis66]. Using Kasteleyn's theory, the authors have proved in [BDT10], Theorem 6, that the law of the low temperature contours on G_n converges to the Gibbs measure $\mathbb{P}^{\infty}_{\text{Ising}}$ on contours of G. In the proof of this theorem, the contours considered have no constraint on their homology class, but a careful analysis is carried out for each of the four terms appearing when expressing the measure in the Fisher dimer setting. Since imposing the homology class of contours, just as in Kasteleyn's theory for bipartite graphs, is just a matter of taking another linear combination of these four terms, the analysis goes through and the limiting measure exists and is *the same* as the non restricted case whatever the topological defect for the Ising model is. As a consequence,

the probability measure on the double critical Ising model converges weakly to the product measure $\mathbb{P}^\infty_{\text{d-Ising}}$.

Since containing $\mathcal{E}$ for XOR (resp. primal) polygon configurations is a local event, *i.e.* a finite union of cylinder events, for the double Ising model (resp. for the bipartite dimer model on G^{Q}), the equality between probabilities of $\mathcal{E}$, which holds on G_n for every n, also holds in the limit: this implies that the law of the polygonal configuration of the critical XOR Ising model on the infinite isoradial graph G is the same as the law of the primal polygon configuration in the critical dimer model on G^{Q}, which ends the proof of the theorem in the case where G is bi-periodic.

Furthermore both measures, for the critical Ising model on G^* [BDT11] and the critical dimer model on G^{Q} [Ken02, dT07a], have the *locality property*: in both models, the probability of a local event only depend on the geometry of the embedding of G in a domain containing the spins or edges involved in this event. This is the crucial property required for the construction of the Gibbs measure of these models on non periodic infinite isoradial graphs [dT07a]. It implies that one can identify the probability of local events on a non periodic graph G with the one on periodic graphs with large fundamental domains coinciding with G on a ball sufficiently large to contain a neighborhood of the region of the graph involved in the event.

As a consequence, probabilities still coincide in the non periodic case: the equality in law holds also when G is not necessarily bi-periodic. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7 Height function on quadri-tilings

Dimer configurations of the quadri-tiling graph G^{Q}, like all bipartite planar dimer models, can be interpreted as random surfaces, via a *height function*. It is the main ingredient to relate the previous results connecting XOR loops and dimers with Wilson's conjecture.

7.1 Definition and properties of the height function

Let us now recall the definition of height function, used in [dT07a]. A dimer configuration M of a planar bipartite graph can be interpreted as a unit flow α_M, flowing by 1 along each matched edge of M, from the white vertex to the black one. It is a function on edges having divergence $+1$ at each white vertex and -1 at each black vertex. Subtracting from α_M another flow with the same divergence at every vertex, yields a divergence-free flow, whose dual is the differential of a function on faces of this graph.

There is a natural candidate for this unit reference flow: since in a dimer configuration there is exactly one dimer incident to every vertex, the sum over all edges incident to any given vertex of the probability that this edge is covered by a dimer, is equal to 1. This means that the flow α_0, flowing by $\mathbb{P}^\infty_{\text{Q}}(e)$[1] from the white vertex to the black one along each edge e of the graph, is a flow with divergence $+1$ (resp. -1) at every white (resp. black) vertex.

---

[1]The graph G^{Q} is isoradial and infinite, and the weights for the quadri-tilings are critical. So in this

The height function h on quadri-tilings is defined as follows. For every dimer configuration M of G^Q, h^M is a function on faces of G^Q, such that for every pair of neighboring faces f and f' of G^Q sharing an edge e, with the additional property that when traversing e from f to f', the black vertex of e is on the left:

$$h^M(f') - h^M(f) = \alpha_M(e) - \alpha_0(e).$$

When faces f and f' are not incident, choose a path $f = f_0, f_1, \cdots, f_n = f'$ in the dual graph joining f and f', then:

$$h^M(f') - h^M(f) = \sum_{i=0}^{n-1}(h^M(f_{i+1}) - h^M(f_i)).$$

This definition is consistent, *i.e* independent of the choice of path from f to f', because the flow $\alpha_M - \alpha_0$ is divergence free; it determines h^M up to a global additive constant, which can be fixed by saying that the height at a particular given face of G^Q is 0. Faces of G^Q are split into three distinct subsets: vertices in the center of faces of $G^\diamond$, vertices of the primal graph G and vertices of the dual graph G^*. We suppose for the sake of definiteness that the face where the height is fixed at 0 corresponds to some particular vertex of G.

Note that another choice of reference unit flow could have been the one coming from the reference dimer configuration M_0, where a white-to-black unit is flowing along all interior edges parallel to edges of G^*. This would have produced a random height function whose restriction to the vertices of G (and to those of G^*) would have coincided with that of this one (up to an additive constant).

Denote by h_V^M (resp. h_{V*}^M) the restriction of h^M to vertices of G (resp. to vertices of G^*).

The next lemma describes possible height changes between pairs of vertices of the primal (resp. dual) graph, incident in the primal (resp. dual) graph. To simplify the picture, we consider primal and dual vertices to be around a rhombus of the diamond graph, see Figure 9.

Lemma 21. *The following assertions are true:*

- *the function h_V^M (resp. h_{M*}^M takes values in $\mathbb{Z}$ (resp. $\mathbb{Z} + \frac{1}{2}$);*

- *The increment of h_V^M (resp. h_{V*}^M) between two neighboring vertices of G (resp. of G^*) is -1, 0, or 1;*

- *The increment of h_V^M (resp. h_{V*}^M) is non zero if and only if the two vertices are separated by an edge of $\mathrm{Poly}_2(M)$ (resp. $\mathrm{Poly}_1(M)$).*

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particular context, we know [Ken02] that the probability of an edge is given by θ/π, where θ is the half-angle of the rhombus containing that edge.
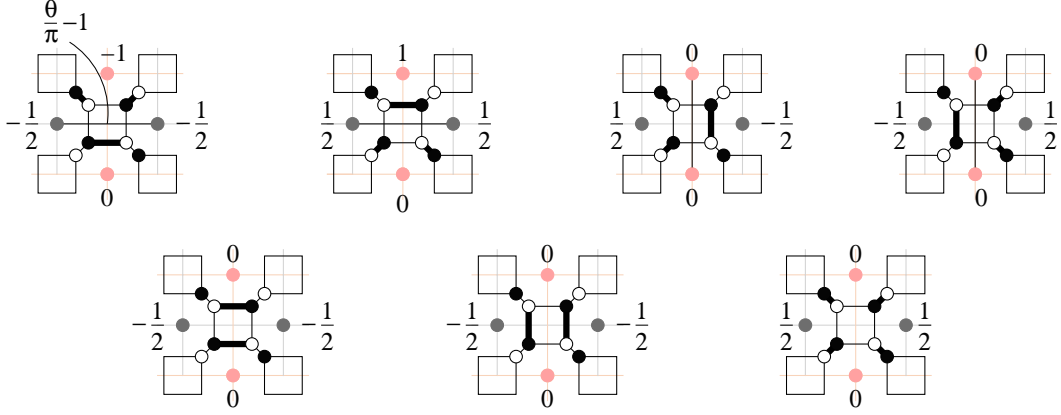
Figure 9: Height changes for the dimer model in a rhombus of the diamond graph.

Proof. Let e_1, e_2 be two interior edges of G^Q, parallel to an edge of G, as in Figure 9. Then, the reference flow α_0 has the same value but opposite direction on these two edges. As a consequence, using the definition of the height function,

$$h^M(v_2^*) - h^M(v_1^*) = \mathbb{I}_{e_1}(M) - \mathbb{P}_Q^\infty(e_1) - \mathbb{I}_{e_2}(M) + \mathbb{P}_Q^\infty(e_1) = \mathbb{I}_{e_1}(M) - \mathbb{I}_{e_2}(M).$$

A similar expression holds for $h^M(v_2) - h^M(v_1)$. This proves that the increment of h^M between two neighboring vertices of G (resp. G^*) is equal to -1, 0, or 1. Because of our convention for the base point, this implies that h^M takes integer values on G. To see that h^M takes half-integer values on G^*, one just has to notice that the reference flow α_0 separating two vertices v (on G) and v^* (on G^*) which are neighbors on $G^\diamond$ is $\frac{\pi/2}{\pi} = \frac{1}{2}$ since the corresponding rhombus in the isoradial graph G^Q is flat. $\square$

The *level lines* of h_V (resp. h_{V^*}) are the set of closed contours on G^* (on resp G) separating clusters of vertices of G (resp. of G^*) where h_V (resp. h_{V^*}) takes the same value.

Returning to the definition of the pair of polygon configurations $\mathrm{Poly}(M)$ assigned to a quadri-tiling M, we immediately obtain the following:

Lemma 22. *Let M be a dimer configuration of G^Q, then level lines of h_V^M, respectively $h_{V^*}^M$, exactly correspond to the polygon configuration $\mathrm{Poly}_1(M)$, respectively $\mathrm{Poly}_2(M)$.*

Note that due to the fact that $\mathrm{Poly}_1(M)$ and $\mathrm{Poly}_2(M)$ do not cross, the increments of h^M along two diagonals of a rhombus cannot be both non zero. As a consequence, on contour lines of h_V^M, $h_{V^*}^M$ is constant.

Combining Lemma 22 with Theorem 20 stating that monochromatic polygon configurations of the XOR Ising model have the same distribution as primal polygon configurations of dimer configurations of G^Q, we obtain one of the main theorem of this paper:

Theorem 23. *Monochromatic polygon configurations of the critical XOR-Ising model have the same distribution as level lines of the restriction to primal vertices of the height function of dimer configurations of G^Q .*

7.2 Wilson's conjecture

In [Wil11], Wilson presented extensive numerical simulations on loops of the critical XOR Ising model on the honeycomb lattice, on the base of which conjectured the following:

Conjecture 1 (Wilson [Wil11])**.** *The scaling limit of the family of loops of the critical XOR Ising model are the level lines of the Gaussian Free Field corresponding to levels that are odd multiples of $\frac{\sqrt{\pi}}{2}$.*

The level lines of the Gaussian Free Field corresponding to levels that are odd multiples of $\lambda = \sqrt{\frac{\pi}{8}}$ form a CLE$_4$ [SS09]. The contour lines of the XOR Ising models are thus conjectured to have the same limiting behavior as the CLE$_4$, except that there are $\sqrt{2}$ fewer loops in the XOR Ising picture. This conjecture is in agreement with predictions of conformal field theory [IR11, PS11].

Theorem 23 can be seen as a proof of a version of Wilson's conjecture in a discrete setting, before passing to the scaling limit, and brings some elements for the complete proof of this conjecture. In particular, it explains the link with the Gaussian Free Field and the factor $\sqrt{2}$, as we will explain now.

For $\varepsilon > 0$, denote by G_ε^Q the embedding of G^Q in the plane where rhombi of G^Q have side length ε. For every dual vertex v in G^{Q^*}, define v^ε the vertex in $G_\varepsilon^{Q^*}$ corresponding to the dual vertex v.

The random height function h can be interpreted on G_ε^Q as a random distribution, *i.e.* a continuous linear form on the set $\mathcal{C}_{0,c}^\infty(\mathbb{R}^2)$ of compactly supported smooth, zero mean functions, denoted by H^ε: for every $\varphi \in \mathcal{C}_{0,c}^\infty(\mathbb{R}^2)$,

$$H^\varepsilon(\varphi) = \sum_{v \in G^{Q^*}} \text{area}(v^\varepsilon) h(v) \varphi(v^\varepsilon),$$

where $\text{area}(v^\varepsilon) = \varepsilon^2 \text{area}(v)$ is the area of the face of G_ε^Q associated to v^ε.

In [dT07b], the second author proved the following convergence result for the height function of the dimer model on G^Q:

Theorem 24 ([dT07b])**.** *As ε goes to 0, the height function on the critical quadri-tilings, as a random distribution, converges in law to $\frac{1}{\sqrt{\pi}}$ times the Gaussian Free field.*

The result also holds for the restriction of h to G (resp. to G^*) as soon as area(v) is replaced by the area of the corresponding face of G (resp. G^*).

As contour lines of the restriction of h to G separate integer values, they can be understood as discrete level lines corresponding to half-integer values. Therefore, it is natural to expect that these contour lines converge to the contour lines of the limiting object, *i.e* to level lines for the Gaussian Free Field with levels $(k+\frac{1}{2})\sqrt{\pi}$, $k \in \mathbb{Z}$, which would prove Wilson's conjecture. Unfortunately, the result for the convergence result of the height function to the Gaussian Free Field is too weak to ensure convergence of contour lines.

The convergence result in the paper [dT07b] applies not only to critical quadri-tilings, but to all bipartite planar dimer models on isoradial graphs with critical weights. It is conjectured that the family of loops obtained by superimposing two independent critical dimer configurations converges to CLE$_4$. This is supported by the fact that each of the dimer configurations can be described by a height function, converging in the scaling limit to $1/\sqrt{\pi}$ times the Gaussian Free Field, the two fields being independent. Dimer loops are the half integer level lines of the difference, which by independence converges (in a weak sense) to $\sqrt{2/\pi}$ times the Gaussian Free Field, and it is known that level lines $(k+1/2)\sqrt{\frac{\pi}{2}}$ of the Gaussian Free Field are a CLE$_4$.

Therefore, the factor $\sqrt{2}$ in Wilson's conjecture corresponds to the fact that contours in the XOR Ising model have to do with contour lines of only one dimer height function, as opposed to two for dimer loops.

As a consequence,

Proposition 25. *Conditional on a proof of the convergence of dimer loops to the* CLE$_4$, *Wilson's conjecture is true.*

A   Some elements of homology theory on surfaces

Here are some general facts about homology theory on surfaces which are useful in the context of this paper. More details can be found in the reference [Ful95, Mau96, Mas91]. We consider Σ to be a compact orientable surface of genus g with boundary $\partial\Sigma$ consisting of p components. The boundary maybe empty, in which case $p = 0$.

We are interested in the *first* homology group H_1, and in the case where the target abelian group is $\mathbb{Z}/2\mathbb{Z}$. The other non trivial homology groups H_0 and H_2 are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, when Σ is connected.

A.1   1-chains and first homology group

A *1-chain* is a 1-dimensional submanifold of Σ. A 1-chain is a *cycle* if its boundary is empty. Among the cycles, there are those which are the boundary of a 2-dimensional submanifold of Σ. Those are simply called *boundaries*. The *first homology group* $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is the free abelian group generated by cycles, modulo boundaries: two cycles γ and γ' represent the same element in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ if their union is the boundary of a 2-dimensional submanifold

of Σ. Note that since the target group is $\mathbb{Z}/2\mathbb{Z}$, we do not need to care about orientations of 1-chains.

A.2   Relative homology

If A is a closed subset of Σ, then one can also consider *homology relative to A*. As above, one defines a notion of cycle and boundary, relative to A this time: a *relative cycle* is a 1-chain whose boundary is in A. A *relative boundary* is a 1-chain in Σ for which there exists a 1-chain in A, such that the union of the two is a boundary in Σ. Then, the *first homology group relative to A*, denoted by $H_1(\Sigma, A; \mathbb{Z}/2\mathbb{Z})$, is the free abelian group generated by relative cycles, modulo relative boundaries.

A particular example of interest is when $A = \partial\Sigma$. Representatives of equivalence classes of $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ are finite unions of cycles and paths attached to components of the boundary.

Note that when $\partial\Sigma$ is empty, the homology of Σ relative to its boundary coincides with the usual homology.

A.3   Explicit bases of homology

The two homology groups $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ turn out to have the same dimension $N = 2g + p - 1$. For each of the groups, a basis can be explicitly given. Label the handles of Σ from 1 to g, and the p components of the boundary from C_0 to C_{p-1}. For $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, choose N cycles $(\boldsymbol{\lambda}_i)_{i=1}^N$ on Σ, as follows:

- for $i \in \{1, \ldots, g\}$, take $\boldsymbol{\lambda}_{2i-1}$ and $\boldsymbol{\lambda}_{2i}$ to be winding around the i-th handle in two transverse directions,

- for $i \in \{1, \ldots, p-1\}$, take $\boldsymbol{\lambda}_{2g+i}$ to be wind around C_i, without crossing $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{2g}$.

Denote by λ_i the homology class of $\boldsymbol{\lambda}_i$. Then, the collection $(\lambda_i)_{i=1}^N$ is a basis of $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$: the homology class of any 1-chain on Σ is homologous to a sum of λ_i's. The first homology group $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^N$: for every $i \in \{1, \cdots, N\}$, the basis element λ_i is mapped to the basis element of $(\mathbb{Z}/2\mathbb{Z})^N$ consisting of 0's and a 1 at position i.

For $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, choose N cycles $(\boldsymbol{\gamma}_i)_{i=1}^N$ on Σ, as follows:

- for $i \in \{1, \ldots, g\}$, take $\boldsymbol{\gamma}_{2i-1} = \boldsymbol{\lambda}_{2i}$, and $\boldsymbol{\gamma}_{2i} = \boldsymbol{\lambda}_{2i-1}$,

- for $i \in \{1, \ldots, p-1\}$, take $\boldsymbol{\gamma}_{2g+i}$ to be a path from C_0 to C_i.

Denote by γ_i the relative homology class of $\boldsymbol{\gamma}_i$. Then, the collection $(\gamma_i)_{i=1}^N$ is a basis for $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$. The group $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ is also isomorphic to $(\mathbb{Z}/2\mathbb{Z})^N$. The two bases $(\lambda_i)_{i=1}^N$ and $(\gamma_i)_{i=1}^N$ are dual to each other as explained in Appendix A.5.

A.4 Representatives of homology classes on graphs

Consider a discretization of the surface Σ by a graph G_Σ, as in Section 2. The embedding of G_Σ on Σ defines a notion of dual graph for G_Σ, denoted by G_Σ^*. Then representatives of any homology class of $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ (resp. any relative homology class of $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$) can be realized as combinatorial paths on G^* (resp. G). Figure 10 provides an example of representatives of the bases $(\lambda_i)_{i=1}^N$ and $(\gamma_i)_{i=1}^N$ defined in Section A.3.
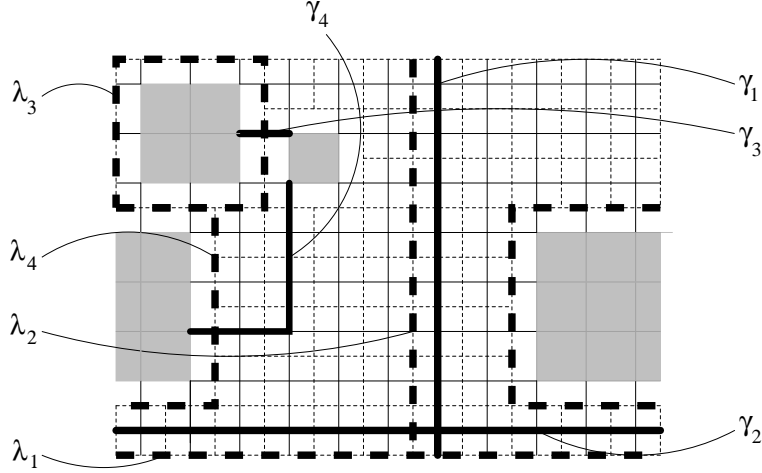

Figure 10: Representatives of a basis $(\lambda_i)_{i=1}^4$ of $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ (dotted lines), and of a basis $(\gamma_i)_{i=1}^4$ of $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ (plain lines).

A.5 Intersection form

There is a natural pairing between $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$, called the *intersection form*:

$$(\cdot|\cdot) : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \times H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

defined as follows. Let $\tau \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and $\epsilon \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ be two homology classes. Take (generic) representatives $\boldsymbol{\tau}$ and $\boldsymbol{\epsilon}$ for two classes τ and ϵ respectively. Then $(\tau|\epsilon)$ is defined as the parity of the number of intersections of $\boldsymbol{\tau}$ and $\boldsymbol{\epsilon}$. This definition does not depend on the choice of representatives.

In the explicit bases of $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ chosen in Appendix A.3, the matrix of the intersection form is the identity. The pairing is thus non degenerate and defines an isomorphism between $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$. This is an explicit realization of the Poincaré-Lefschetz duality [].

A.6 Inclusion, excision and morphisms for homology

Suppose that there exists a larger surface $\tilde{\Sigma}$ containing Σ. Then a 1-chain in Σ is in particular a chain in $\tilde{\Sigma}$, and a boundary in Σ is in particular a boundary in $\tilde{\Sigma}$. This means that the inclusion $\Sigma \subset \tilde{\Sigma}$ induces a morphism

$$\pi_\Sigma : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(\tilde{\Sigma}; \mathbb{Z}/2\mathbb{Z}).$$

The inclusion also induces morphisms for relative homology groups: if the subset $A \subset \Sigma$ is included in a subset $B \subset \tilde{\Sigma}$, then any relative chain (resp. cycle) in Σ relative to A is in particular a relative chain (resp. cycle) in $\tilde{\Sigma}$ relative to B (just by forgetting what is in $B \setminus A$). Therefore, this induces a morphism

$$H_1(\Sigma, A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(\tilde{\Sigma}, B; \mathbb{Z}/2\mathbb{Z}),$$

giving the homology class in $\tilde{\Sigma}$ relative to B of the restriction to $\tilde{\Sigma} \setminus B$ of any representative of an element of $H_1(\Sigma, A; \mathbb{Z}/2\mathbb{Z})$.

Moreover, the *excision theorem* states that if we cut out an open set U from both $\tilde{\Sigma}$ and A, the relative homology groups $H_1(\tilde{\Sigma}, A; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\tilde{\Sigma} \setminus U, A \setminus U; \mathbb{Z}/2\mathbb{Z})$ are isomorphic. In particular, when $U = \Sigma^c = \tilde{\Sigma} \setminus \Sigma$ and $A = \bar{U}$, then the excision theorem states that $H_1(\tilde{\Sigma}, \Sigma^c; \mathbb{Z}/2\mathbb{Z})$ and $H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ are isomorphic.

When composed with the previous morphism given by the inclusion $\Sigma^c \subset \tilde{\Sigma}, \emptyset \subset \tilde{\Sigma}$, it gives the following morphism

$$\Pi_\Sigma : H_1(\tilde{\Sigma}; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z}).$$

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